Nullspace

Previously, we defined a subspace of $\mathbb{R}^n$ to be a non-empty subset of $\mathbb{R}^n$ that is closed under addition and scalar multiplication.
We found that an easy way to define a subspace was as the span of a set of vectors.

Definition: The nullspace of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is the set of all vectors in $\mathbb{R}^n$ whose image under L is the zero vector, $\vec{0}$. We write

$$\mathrm{Null}(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = \vec{0}\}$$

Definition: The nullspace of an $m \times n$ matrix A is

$$\mathrm{Null}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\}$$

When we talk about the nullspace of a matrix, we are thinking of the matrix as a linear mapping, and see that $\mathrm{Null}(L) = \mathrm{Null}([L])$.

Note

The word kernel and the notation $\ker(L)$ or $\ker(A)$ is often used instead of the term nullspace.
The textbook also first defines the nullspace of A as the solution space of A.

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Example

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is in the nullspace of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, since

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 1-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is not in the nullspace of $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, since

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1+1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example

Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$L(x_1, x_2) = (x_1 - x_2, 2x_1 - 2x_2, 3x_1 - 3x_2)$$

Then $(-2, 2)$ is in the nullspace of L, since

$$L(-2, 2) = (-2 - (-2), -4 - (-4), -6 - (-6)) = (0, 0, 0)$$

But $(0, 1)$ is not in the nullspace of L, since

$$L(0, 1) = (-1, -2, -3) \neq (0, 0, 0)$$

Nullspace

Theorem 3.4.1

Let A be an $m \times n$ matrix. Then $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^n$.

Proof

First, we note that $\vec{0} \in \mathrm{Null}(A)$, since $A\vec{0} = \vec{0}$ for any matrix A.
So $\mathrm{Null}(A)$ is non-empty.
Now, suppose $\vec{x}, \vec{y} \in \mathrm{Null}(A)$, and let $t \in \mathbb{R}$.
Then $A\vec{x} = \vec{0} = A\vec{y}$.
Using the linearity properties of matrix multiplication, we have that $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}$.
We also know that $A(t\vec{x}) = tA\vec{x} = t\vec{0} = \vec{0}$.
So $\vec{x} + \vec{y} \in \mathrm{Null}(A)$ and $t\vec{x} \in \mathrm{Null}(A)$.
Thus, we see that $\mathrm{Null}(A)$ is a subspace of $\mathbb{R}^n$. □

Theorem 3.4.1 justifies the use of "space" in the word nullspace.
Since $\mathrm{Null}(L) = \mathrm{Null}([L])$, we also have that $\mathrm{Null}(L)$ is a subspace of $\mathbb{R}^n$.

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Example

Find the nullspace of $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 4 \\ 3 & 9 & 6 \end{bmatrix}$.

Solution

This is the same as finding the general solution to the homogeneous system $A\vec{x} = \vec{0}$, which has coefficient matrix A.

$$\begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 4 \\ 3 & 9 & 6 \end{bmatrix} \begin{matrix} \\ R_2 + R_1 \\ R_3 - 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \begin{matrix} \\ \\ R_3 - 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The last matrix is in RREF, and is equivalent to the system

$$\begin{array}{rcrcl} x_1 & & - 7x_3 & = & 0 \\ & x_2 & + 3x_3 & = & 0 \end{array}$$

Nullspace

Example

Find the nullspace of $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 4 \\ 3 & 9 & 6 \end{bmatrix}$.

Solution

The last matrix is in RREF, and is equivalent to the system

$$\begin{array}{rcrcrcl} x_1 & & & - & 7x_3 & = & 0 \\ & & x_2 & + & 3x_3 & = & 0 \end{array}$$

Replacing the variable x_3 with the parameter t, we get

$$\begin{array}{rcrcrcl} x_1 & & & - & 7t & = & 0 \\ & & x_2 & + & 3t & = & 0 \end{array}$$

So we see that the general solution to the homogeneous system $A\vec{x} = \vec{0}$ is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7t \\ -3t \\ t \end{bmatrix} = t\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}$.

That is, we have $\text{Null}(A) = \text{Span}\left\{\begin{bmatrix} 7 \\ -3 \\ 1 \end{bmatrix}\right\}$.

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Example

Find the nullspace of L, where L is defined by $L(x_1, x_2, x_3) = (x_1 + 2x_2, x_1 - 2x_2 + 4x_3)$.

Solution

We can take the question at face value and look for all solutions to $\begin{bmatrix} x_1 + 2x_2 \\ x_1 - 2x_2 + 4x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is equivalent to the system of linear equations

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & & 0 \\ x_1 & - & 2x_2 & + & 4x_3 & = & 0 \end{array}$$

To solve this system, we need to row reduce the coefficient matrix $\begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & 4 \end{bmatrix}$.

OR

We could use the fact that $\text{Null}(L) = \text{Null}([L])$.
To find $[L]$, we note that $L(1,0,0) = (1,1)$, $L(0,1,0) = (2,-2)$, and $L(0,0,1) = (0,4)$.
Then $[L] = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & 4 \end{bmatrix}$, as we noted earlier.
Then we would want to find the general solution to $[L]\vec{x} = \vec{0}$, which is equivalent to finding the general solution to the system of linear equations given earlier.

So, no matter which way we start the question, we end up looking for the general solution to $[L]\vec{x} = \vec{0}$.

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Example

Find the nullspace of L, where L is defined by $L(x_1, x_2, x_3) = (x_1 + 2x_2, x_1 - 2x_2 + 4x_3)$.

Solution

$$\begin{bmatrix} 1 & 2 & 0 \\ 1 & -2 & 4 \end{bmatrix} \; R_2 - R_1 \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & -4 & 4 \end{bmatrix} \; \tfrac{-1}{4} R_2 \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \end{bmatrix} \; R_1 - 2R_2 \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

This last matrix is in RREF, and is equivalent to the system

$$\begin{array}{ccccccc} x_1 & & & + & 2x_3 & = & 0 \\ & & x_2 & - & x_3 & = & 0 \end{array}$$

Replacing the variable x_3 with the parameter t, we get

$$\begin{array}{ccccccc} x_1 & & & + & 2t & = & 0 \\ & & x_2 & - & t & = & 0 \end{array}$$

So, we see that the general solution to $[L]\vec{x} = \vec{0}$ is $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.

And so we see that $\mathrm{Null}(L) = \mathrm{Span}\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}$.