A Review of Matrices

Definition: A matrix is a rectangular array of numbers. We say that A is an $m \times n$ matrix when A has m rows and n columns, such as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

Definition: Two matrices A and B are equal if and only if they have the same size (that is, the same number of rows and the same number of columns) and their corresponding entries are equal. That is, if $a_{ij} = b_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

We sometimes denote the ij-th entry of a matrix A by $(A)_{ij}$. This is taken to be the same thing as a_{ij}.

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Definition: Let A and B be $m \times n$ matrices. We define addition of matrices by

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

That is, the ij-th entry of $A + B$ is the sum of the ij-th entry of A with the ij-th entry of B.

Definition: Let A be an $m \times n$ matrix, and $t \in \mathbb{R}$ a scalar. We define the scalar multiplication of matrices by

$$(tA)_{ij} = t(A)_{ij}$$

That is, the ij-th entry of tA is t times the ij-th entry of A.

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Theorem 3.1.1

Let A, B, and C be $m \times n$ matrices and let s and t be real scalars. Then

1. $A + B$ is an $m \times n$ matrix — closed under addition
2. $A + B = B + A$ — addition is commutative
3. $(A + B) + C = A + (B + C)$ — addition is associative
4. There exists a matrix, denoted by $0_{m,n}$, such that $A + 0_{m,n} = A$ — zero matrix
5. For every matrix A, there exists an $m \times n$ matrix $(-A)$ such that $A + (-A) = 0_{m,n}$ — additive inverse
6. sA is an $m \times n$ matrix — closed under scalar multiplication
7. $s(tA) = (st)A$ — scalar multiplication is associative
8. $(s + t)A = sA + tA$ — matrix distribution
9. $s(A + B) = sA + sB$ — scalar distribution
10. $1A = A$ — scalar multiplicative identity

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Definition: Let $\mathcal{B} = \{A_1, \ldots, A_k\}$ be a set of $m \times n$ matrices. Then the span of $\mathcal{B}$ is defined as

$$\operatorname{Span} \mathcal{B} = \{t_1 A_1 + \cdots + t_k A_k \mid t_1, \ldots, t_k \in \mathbb{R}\}$$

That is, Span $\mathcal{B}$ is the set of all linear combinations of the matrices in $\mathcal{B}$.

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Example

Determine if $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

Solution

We need to see if there are scalars t_1, t_2, t_3, and t_4 such that

$$t_1 \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} + t_2 \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix} + t_3 \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix} + t_4 \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

Performing the operation on the left side, we see that we need

$$\begin{bmatrix} t_1 - 2t_3 + 2t_4 & t_1 + 3t_2 + 4t_3 + 2t_4 \\ 2t_1 + 4t_2 - 4t_3 - 4t_4 & 2t_1 - 3t_2 - 5t_3 + 3t_4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$$

By the definition of equality, this means we are looking for solutions to the following system of linear equations:

$$\begin{array}{rcrcrcrcr} t_1 & & & - & 2t_3 & + & 2t_4 & = & -1 \\ t_1 & + & 3t_2 & + & 4t_3 & + & 2t_4 & = & 2 \\ 2t_1 & + & 4t_2 & - & 4t_3 & - & 4t_4 & = & 2 \\ 2t_1 & - & 3t_2 & - & 5t_3 & + & 3t_4 & = & 1 \end{array}$$

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Example

Determine if $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

Solution

We row reduce the augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 1 & 3 & 4 & 2 & 2 \\ 2 & 4 & -4 & -4 & 2 \\ 2 & -3 & -5 & 3 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 3 & 6 & 0 & 3 \\ 0 & 4 & 0 & -8 & 4 \\ 0 & -3 & -1 & -1 & 3 \end{array}\right] \begin{array}{l} \frac{1}{3} R_2 \\ \frac{1}{4} R_3 \end{array} \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 0 & -2 & 1 \\ 0 & -3 & -1 & -1 & 3 \end{array}\right] \begin{array}{l} \\ \\ R_3 - R_2 \\ R_4 + 3R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & 5 & -1 & 6 \end{array}\right] \frac{-1}{2} R_3 \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 5 & -1 & 6 \end{array}\right] R_4 - 5R_3 \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -6 & 6 \end{array}\right] \frac{-1}{6} R_4 \sim$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 2 & -1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 - 2R_4 \\ \\ R_3 - R_4 \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 + 2R_3 \\ R_2 - 2R_3 \\ \\ \\ \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

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Example

Determine if $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ is in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

Solution

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

We see from the RREF matrix that $t_1 = 3, t_2 = -1, t_3 = 1, t_4 = -1$ is a solution to our system.
This means that

$$\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix} = 3\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix}$$

Thus, $\begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ **is** in the span of $\left\{ \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 3 \\ 4 & -3 \end{bmatrix}, \begin{bmatrix} -2 & 4 \\ -4 & -5 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ -4 & 3 \end{bmatrix} \right\}$.

A Review of Matrices

Definition: Let $\mathcal{B} = \{A_1, \dots, A_k\}$ be a set of $m \times n$ matrices. Then $\mathcal{B}$ is said to be linearly independent if the only solution to the equation

$$t_1A_1 + \cdots + t_kA_k = 0_{m,n}$$

is the trivial solution $t_1 = \cdots = t_k = 0$. Otherwise, $\mathcal{B}$ is said to be linearly dependent.

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Example

Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ is linearly independent.

Solution

To do this, we need to see how many solutions there are to the equation

$$t_1 \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix} + t_2 \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix} + t_3 \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculations on the left side, we see that this is the same as

$$\begin{bmatrix} t_1 + 8t_3 & 3t_1 - 2t_2 + 6t_3 \\ -t_1 + t_2 + 5t_3 & -3t_1 + 5t_2 - t_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is the same as looking for solutions to the system of homogeneous equations

$$\begin{array}{rcrcrcl} t_1 & & & + & 8t_3 & = & 0 \\ 3t_1 & - & 2t_2 & + & 6t_3 & = & 0 \\ -t_1 & + & t_2 & + & 5t_3 & = & 0 \\ -3t_1 & + & 5t_2 & - & t_3 & = & 0 \end{array}$$

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Example

Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ is linearly independent.

Solution

We row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 0 & 8 \\ 3 & -2 & 6 \\ -1 & 1 & 5 \\ -3 & 5 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 + R_1 \\ R_4 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & -2 & 18 \\ 0 & 1 & 13 \\ 0 & 5 & 25 \end{bmatrix} \begin{matrix} \\ \frac{-1}{2} R_2 \\ \\ \frac{1}{5} R_4 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 1 & 13 \\ 0 & 1 & 5 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 0 & 22 \\ 0 & 0 & 14 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 - \frac{14}{22} R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & -9 \\ 0 & 0 & 22 \\ 0 & 0 & 0 \end{bmatrix}$$

This final matrix is in REF, and so we see that the rank of the coefficient matrix is 3.

Since this is the same as the number of variables, there are no parameters in the general solution to our homogeneous system.

This means that there is only one solution to the system, and we know that this must be the trivial solution.

Therefore, the set $\left\{ \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}, \begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}, \begin{bmatrix} 8 & 6 \\ 5 & 1 \end{bmatrix} \right\}$ **is** linearly independent.

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Example

Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} \right\}$ is linearly independent.

Solution

We need to see how many solutions there are to the equation

$$t_1 \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} + t_2 \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix} + t_3 \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Performing the calculations on the left side, we see that this is the same as

$$\begin{bmatrix} t_1 - t_2 + 2t_3 & t_1 + 2t_2 + 11t_3 \\ 3t_1 - 8t_2 - 9t_3 & -t_1 + 3t_2 + 4t_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

This is the same as looking for solutions to the system of homogeneous equations

$$\begin{array}{rcrcrcl} t_1 & - & t_2 & + & 2t_3 & = & 0 \\ t_1 & + & 2t_2 & + & 11t_3 & = & 0 \\ 3t_1 & - & 8t_2 & - & 9t_3 & = & 0 \\ -t_1 & + & 3t_2 & + & 4t_3 & = & 0 \end{array}$$

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Example

Determine whether or not the set $\left\{ \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} \right\}$ is linearly independent.

Solution

We row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & 11 \\ 3 & -8 & -9 \\ -1 & 3 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 - 3R_1 \\ R_4 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 3 & 9 \\ 0 & -5 & -15 \\ 0 & 2 & 6 \end{bmatrix} \begin{matrix} \\ (1/3)R_2 \\ (-1/5)R_3 \\ (1/2)R_4 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_3 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This final matrix is in REF, so we see that the rank of the coefficient matrix is 2.

Since the number of variables in the system is 3, this means that there is 1 parameter ($3 - 2 = 1$) in the general solution to the system.

Thus, $t_1 = t_2 = t_3 = 0$ is not the only solution to our equation, and this means that $\left\{ \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -8 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 11 \\ -9 & 4 \end{bmatrix} \right\}$ **is** linearly dependent (that is, it is **not** linearly independent).

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