Coordinates With Respect to an Orthonormal Basis

Theorem 7.1.2

If $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, then the i-th coordinate of a vector $\vec{x} \in \mathbb{R}^n$ with respect to $\mathcal{B}$ is

$$b_i = \vec{x} \cdot \vec{v}_i$$

It follows that $\vec{x}$ can be written as

$$\vec{x} = (\vec{x} \cdot \vec{v}_1)\vec{v}_1 + \cdots + (\vec{x} \cdot \vec{v}_n)\vec{v}_n$$

Proof

The proof of this is similar to the proof that orthogonal sets are linearly independent, except that instead of looking at a linear combination that is equal to the zero vector, we now look at a linear combination that is equal to $\vec{x}$. Since $\mathcal{B}$ is a basis for $\mathbb{R}^n$, we know that there are scalars $b_1, \ldots, b_n$ such that

$$b_1\vec{v}_1 + \cdots + b_n\vec{v}_n = \vec{x}$$

Then, for every $1 \le i \le n$, we can take the dot product of $\vec{v}_i$ with both sides of this equation, getting

$$\begin{aligned} (b_1\vec{v}_1 + \cdots + b_n\vec{v}_n) \cdot \vec{v}_i &= \vec{x} \cdot \vec{v}_i \\ (b_1\vec{v}_1) \cdot \vec{v}_i + \cdots + (b_n\vec{v}_n) \cdot \vec{v}_i &= \vec{x} \cdot \vec{v}_i \\ b_1(\vec{v}_1 \cdot \vec{v}_i) + \cdots + b_i(\vec{v}_i \cdot \vec{v}_i) + \cdots + b_n(\vec{v}_n \cdot \vec{v}_i) &= \vec{x} \cdot \vec{v}_i \\ b_1(0) + \cdots + b_i(\|\vec{v}_i\|^2) + \cdots + b_n(0) &= \vec{x} \cdot \vec{v}_i \\ b_i &= \vec{x} \cdot \vec{v}_i \end{aligned}$$

Coordinates With Respect to an Orthonormal Basis

Example

To find the coordinates of $\vec{x} = \begin{bmatrix} -4 \\ -7 \end{bmatrix}$ with respect to the orthonormal basis $\mathcal{B}_1 = \left\{ \begin{bmatrix} 2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix}, \begin{bmatrix} 6/\sqrt{52} \\ 4/\sqrt{52} \end{bmatrix} \right\}$ for $\mathbb{R}^2$, we calculate them individually.

$$\begin{aligned} b_1 &= \begin{bmatrix} -4 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix} = \frac{-8}{\sqrt{13}} + \frac{21}{\sqrt{13}} = \frac{13}{\sqrt{13}} \\ b_2 &= \begin{bmatrix} -4 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} 6/\sqrt{52} \\ 4/\sqrt{52} \end{bmatrix} = \frac{-24}{\sqrt{52}} - \frac{28}{\sqrt{52}} = -\frac{52}{\sqrt{52}} \end{aligned}$$

And so we see that $[\vec{x}]_{\mathcal{B}_1} = \begin{bmatrix} 13/\sqrt{13} \\ -52/\sqrt{52} \end{bmatrix}$

To find the coordinates of $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ with respect to the orthonormal basis $\mathcal{B}_2 = \left\{ \begin{bmatrix} 3/\sqrt{26} \\ 1/\sqrt{26} \\ 4/\sqrt{26} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{78} \\ -7/\sqrt{78} \\ -2/\sqrt{78} \end{bmatrix} \right\}$ for $\mathbb{R}^3$, we again calculate them individually.

Coordinates With Respect to an Orthonormal Basis

$$b_1 = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}3/\sqrt{26}\\1/\sqrt{26}\\4/\sqrt{26}\end{bmatrix} = \frac{3}{\sqrt{26}} + \frac{2}{\sqrt{26}} + \frac{12}{\sqrt{26}} = \frac{17}{\sqrt{26}}$$

$$b_2 = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}-1/\sqrt{3}\\-1/\sqrt{3}\\1/\sqrt{3}\end{bmatrix} = -\frac{1}{\sqrt{3}} - \frac{2}{\sqrt{3}} + \frac{3}{\sqrt{3}} = 0$$

$$b_3 = \begin{bmatrix}1\\2\\3\end{bmatrix} \cdot \begin{bmatrix}5/\sqrt{78}\\-7/\sqrt{78}\\-2/\sqrt{78}\end{bmatrix} = \frac{5}{\sqrt{78}} - \frac{14}{\sqrt{78}} - \frac{6}{\sqrt{78}} = -\frac{15}{\sqrt{78}}$$

And so we see that $[\vec{y}]_{\mathcal{B}_2} = \begin{bmatrix}17/\sqrt{26}\\0\\-15/\sqrt{78}\end{bmatrix}$

Coordinates With Respect to an Orthonormal Basis

Theorem 7.1.a

Let $\vec{x}$ and $\vec{y}$ be vectors in $\mathbb{R}^n$, let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and let $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix}x_1\\\vdots\\x_n\end{bmatrix}$ and $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix}y_1\\\vdots\\y_n\end{bmatrix}$. Then

1. $\vec{x} \cdot \vec{y} = [\vec{x}]_{\mathcal{B}} \cdot [\vec{y}]_{\mathcal{B}}$ and
2. $||\vec{x}|| = ||[\vec{x}]_{\mathcal{B}}||$

Coordinates With Respect to an Orthonormal Basis

Proof

1. We see that

$$\begin{aligned}\vec{x}\cdot\vec{y} &= (x_1\vec{v}_1+\cdots+x_n\vec{v}_n)\cdot(y_1\vec{v}_1+\cdots+y_n\vec{v}_n)\\ &= x_1\vec{v}_1\cdot(y_1\vec{v}_1+\cdots+y_n\vec{v}_n)+\cdots+x_n\vec{v}_n\cdot(y_1\vec{v}_1+\cdots+y_n\vec{v}_n)\\ &= (x_1\vec{v}_1)\cdot(y_1\vec{v}_1)+\cdots+(x_1\vec{v}_1)\cdot(y_n\vec{v}_n)\\ &\qquad+\cdots+(x_n\vec{v}_n)\cdot(y_1\vec{v}_1)+\cdots+(x_n\vec{v}_n)\cdot(y_n\vec{v}_n)\\ &= x_1y_1(\vec{v}_1\cdot\vec{v}_1)+\cdots+(x_1y_n)(\vec{v}_1\cdot\vec{v}_n)+x_2y_1(\vec{v}_2\cdot\vec{v}_1)+\cdots+x_2y_n(\vec{v}_2\cdot\vec{v}_n)\\ &\qquad+\cdots+x_ny_1(\vec{v}_n\cdot\vec{v}_1)+\cdots+x_ny_n(\vec{v}_n\cdot\vec{v}_n)\\ &= x_1y_1+x_2y_2+\cdots+x_ny_n\end{aligned}$$

since $\vec{v}_i\cdot\vec{v}_i=||\vec{v}_i||^2=1^2=1$ and $\vec{v}_i\cdot\vec{v}_j=0$ when $i\neq j$.

And so we have that $\vec{x}\cdot\vec{y}=x_1y_1+x_2y_2+\cdots+x_ny_n=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}\cdot\begin{bmatrix}y_1\\ \vdots\\ y_n\end{bmatrix}=[\vec{x}]_{\mathcal{B}}\cdot[\vec{y}]_{\mathcal{B}}$.

Coordinates With Respect to an Orthonormal Basis

Proof

(2.) follows from (1.), since $\|x\|^2=$

$$\vec{x}\cdot\vec{x}=[\vec{x}]_B\cdot[\vec{x}]_B=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}\cdot\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}=x_1^2+\cdots+x_n^2$$

And so we have $\|x\|=\sqrt{x_1^2+\cdots+x_n^2}=\|[\vec{x}]_B\|$.