Division of Complex Numbers

One of our first uses of complex conjugation will be to help us divide complex numbers. Remember that division is simply an inverse operation for multiplication.
That is, we say that

$$\frac{a}{b} = c \text{ if and only if } a = bc$$

In the complex numbers, we want to notice the following:

$$z_1 = z_2 z_3 \quad \Rightarrow \quad \overline{z_2} z_1 = \overline{z_2} z_2 z_3 \quad \Rightarrow \quad \overline{z_2} z_1 = (x_2^2 + y_2^2) z_3 \quad \Rightarrow \quad \frac{\overline{z_2} z_1}{(x_2^2 + y_2^2)} = z_3$$

And so we can write

$$z_3 = \frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{(x_2^2 + y_2^2)}$$

We expand this formula even further to get the following definition.

Definition: The quotient of two complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is

$$\frac{z_1}{z_2} = \frac{z_1 \overline{z_2}}{z_2 \overline{z_2}} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{y_1 x_2 - x_1 y_2}{x_2^2 + y_2^2} i$$

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I should like to note that I don't think anyone ever bothers to memorize this formula. We simply go through the process of multiplying the top and bottom by the conjugate each time.

Example

$$\begin{aligned} \frac{3-2i}{1+4i} &= \frac{(3-2i)(1-4i)}{(1+4i)(1-4i)} \\ &= \frac{(3)(1) + (3)(-4i) + (-2i)(1) + (-2i)(-4i)}{1^2 + 4^2} \\ &= \frac{(3-8) + (-12-2)i}{1+16} \\ &= -\frac{5}{17} - \frac{14}{17} i \end{aligned}$$