Extending a Linearly Independent Subset to a Basis

Theorem 4.3.b

If $\mathcal{T} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set, and if $\mathbf{w} \notin \operatorname{Span} \mathcal{T}$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}\}$ is also a linearly independent set.

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Proof

Suppose that $\mathcal{T} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a linearly independent set, and that $\mathbf{w} \notin \operatorname{Span} \mathcal{T}$. To see that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}\}$ is linearly independent, we will look for solutions to the equation

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{w} = \mathbf{0}$$

Suppose, by way of contradiction, that $t_{k+1} \neq 0$. Then we can divide by t_{k+1} and get

$$\frac{t_1}{t_{k+1}}\mathbf{v}_1 + \cdots + \frac{t_k}{t_{k+1}}\mathbf{v}_k + \mathbf{w} = \mathbf{0}$$

which means that

$$-\frac{t_1}{t_{k+1}}\mathbf{v}_1 - \cdots - \frac{t_k}{t_{k+1}}\mathbf{v}_k = \mathbf{w}$$

But this means we can write $\mathbf{w}$ as a linear combination of the vectors in $\mathcal{T}$, which contradicts our choice of $\mathbf{w}$ as not being in the span of $\mathcal{T}$. From this contradiction, we know that $t_{k+1} = 0$. And this turns our linear independence equation into

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + 0\mathbf{w} = \mathbf{0}$$

which is the same as

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k = \mathbf{0}$$

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Proof

And since $\mathcal{T}$ is linearly independent, we know that the only solution to this equation is $t_1 = \ldots = t_k = 0$.
As such, we have shown that the only solution to the equation

$$t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k + t_{k+1}\mathbf{w} = \mathbf{0}$$

is $t_1 = \ldots = t_k = t_{k+1} = 0$.
And this means that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{w}\}$ is linearly independent.

Now, the fact that our vector spaces have a finite basis does more than guarantee that our expansion process will come to an end. It actually tells us **when** our process will end. Because our basis will need to have exactly $\mathbf{dim}\,\mathbb{V}$ elements in it. We see how to use this fact in the following example.

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Example

(a) Produce a basis $\mathcal{B}$ for the plane $\mathcal{P}$ in $\mathbb{R}^3$ with equation $2x_1 + 4x_2 - x_3 = 0$, and
(b) extend the basis $\mathcal{B}$ to a basis $\mathcal{C}$ for $\mathbb{R}^3$.

We already know that the dimension of any plane in $\mathbb{R}^3$ is 2, so to find a basis $\mathcal{B}$ for $\mathcal{P}$, we simply need to find two linearly independent vectors on our plane.

But a set of two vectors is linearly independent whenever they are not a scalar multiple of each other.
So we can quickly note that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

are two vectors that satisfy

$$2x_1 + 4x_2 - x_3 = 0$$

that are not scalar multiples of each other.
And thus,

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\mathcal{P}$.

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Example

(a) Produce a basis $\mathcal{B}$ for the plane $\mathcal{P}$ in $\mathbb{R}^3$ with equation $2x_1 + 4x_2 - x_3 = 0$, and
(b) extend the basis $\mathcal{B}$ to a basis $\mathcal{C}$ for $\mathbb{R}^3$.

Now, we know that the dimension of $\mathbb{R}^3$ is 3, so to extend $\mathcal{B}$ to a basis $\mathcal{C}$ of $\mathbb{R}^3$, we simply need to find one vector not in the span of $\mathcal{B}$.
But the span of $\mathcal{B}$ is precisely the vectors on the plane $\mathcal{P}$. So this means we are looking for any vector not on the plane.
That is, we are looking for any vector that does **not** satisfy $2x_1 + 4x_2 - x_3 = 0$.

The vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ quickly comes to mind as such a vector.
And so we have that

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}_3$

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Theorem 4.3.4

Let $\mathbb{V}$ be an n-dimensional vector space. Then
(1) A set of more than n vectors in $\mathbb{V}$ must be linearly dependent.
(2) A set of fewer than n vectors cannot span $\mathbb{V}$.
(3) A set with n elements of $\mathbb{V}$ is a spanning set for $\mathbb{V}$ if and only if it is linearly independent.

(3) will be referred to as the "two out of three rule"
Three features a basis must have are: spanning, linear independence, n elements.