Orthogonal Complement

In theory, finding an orthonormal basis is easy. Start with one vector, add a vector that is orthogonal, and then add another that is orthogonal to the first two. Problems arise when dealing with very large spaces.

Definition: Let $\mathbb{S}$ be a subspace of $\mathbb{R}^n$. We shall say that a vector $\vec{x}$ is orthogonal to $\mathbb{S}$ if

$$\vec{x} \cdot \vec{s} = 0 \text{ for all } \vec{s} \in \mathbb{S}$$

Example

Let $\mathbb{S} = \text{Span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$.

Then $\begin{bmatrix}0\\1\\0\\0\end{bmatrix}$ is orthogonal to $\mathbb{S}$, because given any element $a\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\0\\0\\1\end{bmatrix} = \begin{bmatrix}a\\0\\0\\b\end{bmatrix}$ of $\text{Span}\left\{\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\end{bmatrix}\right\}$, we see that $\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}a\\0\\0\\b\end{bmatrix} = 0$. We also see that $\begin{bmatrix}0\\3\\0\\0\end{bmatrix}$ is orthogonal to $\mathbb{S}$, since

$$\begin{bmatrix}0\\3\\0\\0\end{bmatrix} \cdot \begin{bmatrix}a\\0\\0\\b\end{bmatrix} = 3\begin{bmatrix}0\\1\\0\\0\end{bmatrix} \cdot \begin{bmatrix}a\\0\\0\\b\end{bmatrix} = 3(0) = 0$$

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It is also easy to notice that $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ is orthogonal to $\mathbb{S}$, since we also have that $\begin{bmatrix}0\\0\\1\\0\end{bmatrix} \cdot \begin{bmatrix}a\\0\\0\\b\end{bmatrix} = 0$.

But $\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ is not orthogonal to $\mathbb{S}$, since $\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$ is an element of $\mathbb{S}$, but $\begin{bmatrix}1\\1\\1\\1\end{bmatrix} \cdot \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = 1 \neq 0$.

Note that in order to show that $\vec{x}$ is orthogonal to a subspace $\mathbb{S}$, we only need to show that $\vec{x}$ is orthogonal to the basis vectors for $\mathbb{S}$.

If $\mathcal{A} = \{\vec{a}_1, \ldots, \vec{a}_k\}$ is a spanning set for $\mathbb{S}$, then every element $\vec{s}$ of $\mathbb{S}$ can be written as $s_1\vec{a}_1 + \cdots + s_k\vec{a}_k$ for some $s_1, \ldots, s_k \in \mathbb{R}$.

And if $\vec{x}$ is orthogonal every $\vec{a}_i$ for $1 \leq i \leq k$, then we have the following:

$$\begin{aligned}\vec{x} \cdot \vec{s} &= \vec{x} \cdot (s_1\vec{a}_1 + \cdots + s_k\vec{a}_k)\\ &= (\vec{x} \cdot s_1\vec{a}_1) + \cdots + (\vec{x} \cdot s_k\vec{a}_k)\\ &= s_1(\vec{x} \cdot \vec{a}_1) + \cdots + s_k(\vec{x} \cdot \vec{a}_k)\\ &= 0\end{aligned}$$

If $\vec{x}$ is orthogonal to $\mathbb{S}$, then $t\vec{x}$ is orthogonal to $\mathbb{S}$ for all scalars $t \in \mathbb{R}$, since $(t\vec{x}) \cdot \vec{s} = t(\vec{x} \cdot \vec{s}) = t(0) = 0$. Also, if both $\vec{x}$ and $\vec{y}$ are orthogonal to $\mathbb{S}$, then so is $\vec{x} + \vec{y}$, since $(\vec{x} + \vec{y}) \cdot \vec{s} = (\vec{x} \cdot \vec{s}) + (\vec{y} \cdot \vec{s}) = 0 + 0 = 0$. Since $\vec{0} \cdot \vec{v} = 0$ for any vector $\vec{v}$, then we have shown that the set of all vectors orthogonal to $\mathbb{S}$ is never the empty set. And this means that we have shown that the set of all vectors orthogonal to $\mathbb{S}$ is itself a subspace of $\mathbb{R}^n$, since it is a non-empty subset of $\mathbb{R}^n$ that is closed under addition and scalar multiplication.

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Definition: We call the set of all vectors orthogonal to $\mathbb{S}$ the orthogonal complement of $\mathbb{S}$ and denote it $\mathbb{S}^\perp$. That is

$$\mathbb{S}^\perp = \{\vec{x} \in \mathbb{R}^n \mid \vec{x} \cdot \vec{s} = 0 \text{ for all } \vec{s} \in \mathbb{S}\}$$

Example

Find a basis for $\mathbb{S}^\perp$, where $\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\2\\0\\1\end{bmatrix}, \begin{bmatrix}3\\6\\1\\4\end{bmatrix}\right\}$.

A vector is orthogonal to $\mathbb{S}$ if it is orthogonal to the vectors in its spanning set, so we are looking for vectors $\vec{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}$ such that $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \cdot \begin{bmatrix}1\\2\\0\\1\end{bmatrix} = 0$ and $\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} \cdot \begin{bmatrix}3\\6\\1\\4\end{bmatrix} = 0$. This is the same as looking for solutions to the following system:

$$\begin{array}{rrrrl} x_1 & +2x_2 & & +x_4 & = 0 \\ 3x_1 & +6x_2 & +x_3 & +4x_4 & = 0 \end{array}$$

To solve this homogeneous system, we row reduce its coefficient matrix:

$$\begin{bmatrix}1 & 2 & 0 & 1\\3 & 6 & 1 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 0 & 1\\0 & 0 & 1 & 1\end{bmatrix}$$

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From our RREF matrix, we see that our system is equivalent to

$$\begin{array}{rrrrl} x_1 & +2x_2 & & +x_4 & = 0 \\ & & x_3 & +x_4 & = 0 \end{array}$$

Replacing the variable x_2 with the parameter s and the variable x_4 with the parameter t, we get that the general solution to this system is

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-2s-t\\s\\-t\\t\end{bmatrix} = s\begin{bmatrix}-2\\1\\0\\0\end{bmatrix} + t\begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}$$

The general solution to our system is a list of all the vectors $\vec{x}$ that are orthogonal to $\mathbb{S}$, so we see that $\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}\right\}$ is a spanning set for $\mathbb{S}^\perp$.

Moreover, these vectors are not a scalar multiple of each other, and thus are linearly independent, so we have that $\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\-1\\1\end{bmatrix}\right\}$ is a basis for $\mathbb{S}^\perp$.

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Theorem 7.2.1

Let $\mathbb{S}$ be a k-dimensional subspace of $\mathbb{R}^n$. Then

1. $\mathbb{S} \cap \mathbb{S}^\perp = \{\vec{0}\}$
2. $\dim(\mathbb{S}^\perp) = n - k$
3. If $\{\vec{v}_1, \dots, \vec{v}_k\}$ is an orthonormal basis for $\mathbb{S}$ and $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{S}^\perp$, then $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$

Proof

To see that $\mathbb{S} \cap \mathbb{S}^\perp = \{\vec{0}\}$, let $\vec{x} \in \mathbb{S} \cap \mathbb{S}^\perp$.
Then $\vec{x}$ is an element of $\mathbb{S}^\perp$, so $\vec{x}$ is orthogonal to every element of $\mathbb{S}$.
But we also have that $\vec{x}$ is an element of $\mathbb{S}$, so this means that $\vec{x}$ is orthogonal to itself. That is, $\vec{x} \cdot \vec{x} = 0$, which means that $\vec{x} = \vec{0}$.
Next, to see that $\dim(\mathbb{S}^\perp) = n - k$, let A be the matrix whose rows are the basis vectors of $\mathbb{S}$. Then A is a $k \times n$ matrix, and $\mathbb{S}$ is the rowspace of A. This means that the rank of A is the same as the dimension of $\mathbb{S}$, so $\text{rank}(A) = k$.
But we also have that $\mathbb{S}^\perp$ is the nullspace of A, and thus the dimension of $\mathbb{S}^\perp$ is the nullity of A. By the Rank Theorem, we know that $\text{rank}(A) + \text{nullity}(A) = n$, so the $\dim(\mathbb{S}^\perp) = \text{nullity}(A) = n - \text{rank}(A) = n - k$.

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Finally, to see that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthonormal basis for $\mathbb{R}^n$, remember that we, in fact, only need to show that $\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_n\}$ is an orthonormal set (as it will then automatically be a basis). That means we need to show that $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$. We will break this into four different scenarios:

(a) $1 \leq i, j \leq k$. Then both $\vec{v}_i$ and $\vec{v}_j$ are in $\{\vec{v}_1, \dots, \vec{v}_k\}$, which is an orthonormal set, so we know that $\vec{v}_i \cdot \vec{v}_j = 0$.

(b) $1 \leq i \leq k$ and $k + 1 \leq j \leq n$. Then $\vec{v}_i \in \mathbb{S}$ and $\vec{v}_j \in \mathbb{S}^\perp$, so by the definition of $\mathbb{S}^\perp$ we know that $\vec{v}_i \cdot \vec{v}_j = 0$.

(c) $1 \leq j \leq k$ and $k + 1 \leq i \leq n$. Then $\vec{v}_j \in \mathbb{S}$ and $\vec{v}_i \in \mathbb{S}^\perp$, so by the definition of $\mathbb{S}^\perp$ we know that $\vec{v}_i \cdot \vec{v}_j = 0$.

(d) $k + 1 \leq i, j \leq n$. Then both $\vec{v}_i$ and $\vec{v}_j$ are in $\{\vec{v}_{k+1}, \dots, \vec{v}_n\}$, which is an orthonormal set, so we know that $\vec{v}_i \cdot \vec{v}_j = 0$.