Orthogonality in $\mathbb{C}^n$

Now that we have the inner product correctly defined, we can define orthogonality for complex vector spaces.

Definition: Let $\mathbb{V}$ be an inner product space over $\mathbb{C}$, with inner product $\langle\ ,\ \rangle$. Then two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{V}$ are said to be orthogonal if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$. The set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ in $\mathbb{V}$ is said to be orthogonal if $\langle \mathbf{v}_j, \mathbf{v}_k \rangle = 0$ for all $j \neq k$. If we also have $\langle \mathbf{v}_j, \mathbf{v}_j \rangle = 1$ for all $1 \leq j \leq n$, then the set is orthonormal.

Example

The vectors $\vec{u}_1 = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 1+i \\ 1-i \\ 0 \end{bmatrix}$ are orthogonal, since

$$\begin{aligned} \langle \vec{u}_1, \vec{u}_2 \rangle &= \vec{u}_1 \cdot \overline{\vec{u}_2} = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix} \cdot \begin{bmatrix} 1-i \\ 1+i \\ 0 \end{bmatrix} \\ &= 1 - i + i + i^2 + 0 = 0 \end{aligned}$$

The vectors $\vec{v}_1 = \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} i \\ 1 \\ i \end{bmatrix}$ are not orthogonal, since

$$\begin{aligned} \langle \vec{v}_1, \vec{v}_2 \rangle &= \vec{v}_1 \cdot \overline{\vec{v}_2} = \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix} \cdot \begin{bmatrix} -i \\ 1 \\ -i \end{bmatrix} \\ &= -i - i^2 + 1 - i + 2i^2 = -2i \neq 0 \end{aligned}$$

Orthogonality in $\mathbb{C}^n$

Definition: Let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthogonal basis for a subspace $\mathbb{S}$ of $\mathbb{C}^n$. Then the projection of $\vec{z} \in \mathbb{C}^n$ onto $\mathbb{S}$ is given by

$$\text{proj}_{\mathbb{S}} \vec{z} = \frac{\langle \vec{z}, \vec{v}_1 \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \cdots + \frac{\langle \vec{z}, \vec{v}_k \rangle}{\langle \vec{v}_k, \vec{v}_k \rangle} \vec{v}_k$$

While this is the same definition we've always used, it is important to remember that our inner product is no longer symmetric, so we must make sure we take our inner product in the correct order. If we use $\langle \vec{v}_1, \vec{z} \rangle$ instead of $\langle \vec{z}, \vec{v}_1 \rangle$, we'll end up with the wrong answer.

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Example

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 1+i \\ 1-i \\ 0 \end{bmatrix}$ be as in the previous example. Then we already know that $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for $\mathbb{S} = \operatorname{Span}\{\vec{u}_1, \vec{u}_2\}$. Let $\vec{z} = \begin{bmatrix} 1+i \\ 1+2i \\ 1+3i \end{bmatrix}$, and let's find $\operatorname{proj}_{\mathbb{S}}\vec{z}$. From the definition, we see that

$$\operatorname{proj}_{\mathbb{S}}\vec{z} = \frac{\langle \vec{z}, \vec{u}_1\rangle}{\langle \vec{u}_1, \vec{u}_1\rangle}\vec{u}_1 + \frac{\langle \vec{z}, \vec{u}_2\rangle}{\langle \vec{u}_2, \vec{u}_2\rangle}\vec{u}_2$$

Let's compute all the inner products:

$$\langle \vec{z}, \vec{u}_1\rangle = \vec{z}\cdot\overline{\vec{u}_1} = \begin{bmatrix} 1+i \\ 1+2i \\ 1+3i \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -i \\ 2-i \end{bmatrix} = 1+i-i-2i^2+2-i+6i-3i^2 = 8+5i$$

$$\langle \vec{u}_1, \vec{u}_1\rangle = 1^2+1^2+2^2+1^2 = 7$$

$$\langle \vec{z}, \vec{u}_2\rangle = \vec{z}\cdot\overline{\vec{u}_2} = \begin{bmatrix} 1+i \\ 1+2i \\ 1+3i \end{bmatrix} \cdot \begin{bmatrix} 1-i \\ 1+i \\ 0 \end{bmatrix} = 1-i+i-i^2+1+i+2i+2i^2+0 = 1+3i$$

$$\langle \vec{u}_2, \vec{u}_2\rangle = 1^2+1^2+1^2+(-1)^2 = 4$$

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Example

So we see that

$$\begin{aligned}
\operatorname{proj}_{\mathbb{S}}\vec{z} &= \frac{8+5i}{7}\begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix} + \frac{1+3i}{4}\begin{bmatrix} 1+i \\ 1-i \\ 0 \end{bmatrix} \\
&= \frac{1}{28}\left(4(8+5i)\begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix} + 7(1+3i)\begin{bmatrix} 1+i \\ 1-i \\ 0 \end{bmatrix}\right) \\
&= \frac{1}{28}\begin{bmatrix} 32+20i+7+7i+21i+21i^2 \\ 32i+20i^2+7-7i+21i-21i^2 \\ 64+32i+40i+20i^2+0 \end{bmatrix} \\
&= \frac{1}{28}\begin{bmatrix} 18+48i \\ 8+46i \\ 44+72i \end{bmatrix} \\
&= \begin{bmatrix} \frac{9}{14}+\left(\frac{12}{7}\right)i \\ \frac{2}{7}+\left(\frac{23}{14}\right)i \\ \frac{11}{7}+\left(\frac{18}{7}\right)i \end{bmatrix}
\end{aligned}$$

And since the definition of $\operatorname{perp}_{\mathbb{S}}\vec{z}$ and the Gram-Schmidt procedure are all based on the definition of $\operatorname{proj}_{\mathbb{S}}$, these also work the same as they did in $\mathbb{R}^n$.

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Example

Let $\vec{v}_1 = \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} i \\ 1 \\ i \end{bmatrix}$, as in our previous example. Since $\vec{v}_1$ and $\vec{v}_2$ are not orthogonal, the set $\{\vec{v}_1, \vec{v}_2\}$ does not form an orthogonal basis for $\mathbb{S} = \mathbf{Span}\{\vec{v}_1, \vec{v}_2\}$. It is a spanning set, though, so we can use it in the Gram-Schmidt procedure to find an orthogonal basis for $\mathbb{S}$. We start by setting $\vec{w}_1 = \vec{v}_1$ and $\mathbb{S}_1 = \mathbf{Span}\{\vec{w}_1\}$. Then

$$\begin{aligned} \vec{w}_2 &= \mathbf{perp}_{\mathbb{S}_1} \vec{v}_2 \\ &= \vec{v}_2 - \mathbf{proj}_{\mathbb{S}_1} \vec{v}_2 \\ &= \vec{v}_2 - \frac{\langle \vec{v}_2, \vec{w}_1 \rangle}{\langle \vec{w}_1, \vec{w}_1 \rangle} \vec{w}_1 \end{aligned}$$

Let's compute the inner products we need:

$$\langle \vec{v}_2, \vec{w}_1 \rangle = \langle \vec{v}_2, \vec{v}_1 \rangle = \overline{\langle \vec{v}_1, \vec{v}_2 \rangle} = \overline{-2i} = 2i$$
$$\langle \vec{w}_1, \vec{w}_1 \rangle = 1^2 + 1^2 + 1^2 + (-1)^2 + (-2)^2 = 8$$

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Example

So we have that

$$\begin{aligned} \vec{w}_2 &= \begin{bmatrix} i \\ 1 \\ i \end{bmatrix} - \left(\frac{2i}{8}\right) \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix} \\ &= \begin{bmatrix} i - \frac{1}{4} i \left(\frac{1}{4} i^2\right) \\ 1 - \frac{1}{4} i + \frac{1}{4} i^2 \\ i + \frac{1}{2} i^2 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{4} + \frac{3}{4} i \\ \frac{3}{4} - \frac{1}{4} i \\ \frac{-1}{2} + i \end{bmatrix} \end{aligned}$$

As before, we can pull out the scalar $\frac{1}{4}$ from our $\vec{w}_2$ and still have an orthogonal basis vector. And so, by the Gram-Schmidt procedure, we have that $\mathcal{B} = \left\{ \begin{bmatrix} 1+i \\ 1-i \\ -2i \end{bmatrix}, \begin{bmatrix} 1+3i \\ 3-i \\ -2+4i \end{bmatrix} \right\}$ is an orthogonal basis for $\mathcal{S}$.