Orthonormal Bases

By its definition, a basis is a linearly independent spanning set, but the reason we wanted those features is because we wanted to be able to uniquely write every vector as a linear combination of the basis vectors.
That is, we wanted to be able to assign coordinates based on our basis.
Once we have coordinates, we can use all of our results from $\mathbb{R}^n$ (including finding a matrix to represent any linear transformation).
It would be nice if coordinates were easy to find. This is a benefit of the standard basis.
First, I want to point out that from this point forward, we will again focus our attention purely on $\mathbb{R}^n$, since we now know that we can extend these results to any finite dimensional vector space.
When we are focused on $\mathbb{R}^n$, we can turn our attention to the dot product.

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Definition: Let $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. Then the dot product of $\vec{x}$ and $\vec{y}$ is

$$\vec{x} \cdot \vec{y} = x_1 y_1 + \cdots + x_n y_n$$

We also want to recall that two vectors $\vec{x}$ and $\vec{y}$ are orthogonal if $\vec{x} \cdot \vec{y} = 0$. We now wish to extend this notion of orthogonality to sets of vectors.

Definition: A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is orthogonal if $v_i \cdot v_j = 0$ whenever $i \neq j$.

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Example

The set $\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ 4 \end{bmatrix} \right\}$ is orthogonal, because

$$\begin{bmatrix} 2 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} = (2)(6) + (-3)(4) = 12 - 12 = 0$$

The set $\left\{ \begin{bmatrix} 7 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$ is not orthogonal, because

$$\begin{bmatrix} 7 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \end{bmatrix} = (7)(-1) + (4)(2) = -7 + 8 = 1 \neq 0$$

The set $\left\{ \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -7 \\ -2 \end{bmatrix} \right\}$ is orthogonal, because

$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = -3 - 1 + 4 = 0$$
$$\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -7 \\ -2 \end{bmatrix} = 15 - 7 - 8 = 0$$
$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ -7 \\ -2 \end{bmatrix} = -5 + 7 - 2 = 0$$

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Example

The set $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ -3 \\ -12 \\ -2 \end{bmatrix}, \begin{bmatrix} 6 \\ -5 \\ -9 \\ 4 \end{bmatrix} \right\}$ is not orthogonal, because

$$\begin{bmatrix} 8 \\ -3 \\ -12 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -5 \\ -9 \\ 4 \end{bmatrix} = 48 + 15 + 108 - 8 = 163 \neq 0.$$

Example

The standard basis $\{\vec{e}_1, \ldots, \vec{e}_n\}$ for $\mathbb{R}^n$ is orthogonal, since $\vec{e}_i \cdot \vec{e}_j = 0$ if $i \neq j$.

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Theorem 7.1.1

If $\{v_1, \ldots, v_k\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$, it is linearly independent.

Proof

Note first that we need the vectors to be non-zero, since any set containing the zero vector is linearly dependent.
And so, let's assume that $\{v_1, \ldots, v_k\}$ is an orthogonal set of non-zero vectors in $\mathbb{R}^n$.
To see that $\{v_1, \ldots, v_k\}$ is linearly independent, let's assume we have scalars $c_1, \ldots, c_k$ such that

$$c_1 v_1 + \cdots + c_k v_k = \vec{0}$$

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Then, for every $1 \leq i \leq k$, we can take the dot product of v_i with both sides of this equation, getting

$$\begin{aligned}
(c_1 v_1 + \cdots + c_k v_k) \cdot v_i &= \vec{0} \cdot v_i \\
(c_1 v_1) \cdot v_i + \cdots + (c_k v_k) \cdot v_i &= 0 \\
c_1(v_1 \cdot v_i) + \cdots + c_i(v_i \cdot v_i) + \cdots + c_k(v_k \cdot v_i) &= 0 \\
c_1(0) + \cdots + c_i(\|v_i\|^2) + \cdots + c_k(0) &= 0 \\
c_i(\|v_i\|^2) &= 0
\end{aligned}$$

Since we know $v_i \neq \vec{0}$, we know that $\|v_i\|^2 \neq 0$, which means we can divide by it and get that $c_i = 0$.
And since this is true for all $1 \leq i \leq k$, we have shown that all $c_i = 0$ for $1 \leq i \leq k$, which means that $\{v_1, \ldots, v_k\}$ is linearly independent.

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In addition to being orthogonal, the standard basis has one other nice property: the length (or norm) of each of the vectors is 1.
If we add that requirement to a general orthogonal set, we say that the set is orthonormal.

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Definition: A set $\{v_1 \ \dots, v_k\}$ of vectors in $\mathbb{R}^n$ is orthonormal if it is orthogonal and each vector v_i is a unit vector (that is, each vector is normalized).

Note that, since the zero vector does not have length 1, it can never be in an orthonormal set. So, using Theorem 7.1.1, we see that all orthonormal sets are linearly independent.

Example

The set $\left\{ \begin{bmatrix} 2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix}, \begin{bmatrix} 6/\sqrt{52} \\ 4/\sqrt{52} \end{bmatrix} \right\}$ is an orthonormal set, since

$\left\| \begin{bmatrix} 2/\sqrt{13} \\ -3/\sqrt{13} \end{bmatrix} \right\| = \sqrt{4/13 + 9/13} = 1$

and $\left\| \begin{bmatrix} 6/\sqrt{52} \\ 4/\sqrt{52} \end{bmatrix} \right\| = \sqrt{36/52 + 16/52} = 1$

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The set $\left\{ \begin{bmatrix} 3/\sqrt{26} \\ 1/\sqrt{26} \\ 4/\sqrt{26} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 5/\sqrt{78} \\ -7/\sqrt{78} \\ -2/\sqrt{78} \end{bmatrix} \right\}$ is orthonormal, since

$$\begin{bmatrix} 3/\sqrt{26} \\ 1/\sqrt{26} \\ 4/\sqrt{26} \end{bmatrix} \cdot \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = -\frac{3}{\sqrt{78}} - \frac{1}{\sqrt{78}} + \frac{4}{\sqrt{78}} = 0$$

$$\begin{bmatrix} 3/\sqrt{26} \\ 1/\sqrt{26} \\ 4/\sqrt{26} \end{bmatrix} \cdot \begin{bmatrix} 5/\sqrt{78} \\ -7/\sqrt{78} \\ -2/\sqrt{78} \end{bmatrix} = \frac{15}{\sqrt{2028}} - \frac{7}{\sqrt{2028}} - \frac{8}{\sqrt{2028}} = 0$$

$$\begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} 5/\sqrt{78} \\ -7/\sqrt{78} \\ -2/\sqrt{78} \end{bmatrix} = -\frac{5}{\sqrt{234}} + \frac{7}{\sqrt{234}} - \frac{2}{\sqrt{234}} = 0$$

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$$\left\| \begin{bmatrix} 3/\sqrt{26} \\ 1/\sqrt{26} \\ 4/\sqrt{26} \end{bmatrix} \right\| = \sqrt{\frac{9}{26} + \frac{1}{26} + \frac{16}{26}} = 1$$

$$\left\| \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\| = \sqrt{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} = 1$$

$$\left\| \begin{bmatrix} 5/\sqrt{78} \\ -7/\sqrt{78} \\ -2/\sqrt{78} \end{bmatrix} \right\| = \sqrt{\frac{25}{78} + \frac{49}{78} + \frac{4}{78}} = 1$$