Properties of Complex Inner Product Spaces

When doing the problems assigned for the previous lecture, you hopefully noticed that the standard inner product for complex numbers is not symmetrical (that is, that $\langle \vec{z}, \vec{w} \rangle \neq \langle \vec{w}, \vec{z} \rangle$). So, right away we know that our definition of an inner product will have to be different than the one we used for the reals.

But, hopefully you also noticed that $\langle \vec{z}, \vec{w} \rangle = \overline{\langle \vec{w}, \vec{z} \rangle}$, so this is yet another case where we need to introduce conjugation to extend a result from the reals to the complex numbers. Bilinearity also needs some adjustments in the complex numbers.

Let's take a look at the definition of a (generic) inner product on $\mathbb{C}^n$.

Properties of Complex Inner Product Spaces

Definition: Let $\mathbb{V}$ be a vector space over $\mathbb{C}$. A complex inner product on $\mathbb{V}$ is a function $\langle \, , \, \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{C}$ such that

1. For all $\mathbf{z} \in \mathbb{V}$, we have that $\langle \mathbf{z}, \mathbf{z} \rangle$ is a non-negative real number, and $\langle \mathbf{z}, \mathbf{z} \rangle = 0$ if and only if $\mathbf{z} = \mathbf{0}$.
2. For all $\mathbf{w}, \mathbf{z} \in \mathbb{V}$, $\langle \mathbf{z}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{z} \rangle}$
3. For all $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{V}$ and all $\alpha \in \mathbb{C}$ we have

   i. $\langle \mathbf{v} + \mathbf{z}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{z}, \mathbf{w} \rangle$
   ii. $\langle \mathbf{z}, \mathbf{w} + \mathbf{u} \rangle = \langle \mathbf{z}, \mathbf{w} \rangle + \langle \mathbf{z}, \mathbf{u} \rangle$
   iii. $\langle \alpha\mathbf{z}, \mathbf{w} \rangle = \alpha \langle \mathbf{z}, \mathbf{w} \rangle$
   iv. $\langle \mathbf{z}, \alpha\mathbf{w} \rangle = \overline{\alpha} \langle \mathbf{z}, \mathbf{w} \rangle$

Property 1 is still the same as in $\mathbb{R}^n$, and is still referred to as being "positive definite".

Property 2 is known as the **Hermitian** property of the inner product (instead of the symmetric property). Because the complex inner product is not symmetric, we cannot find a simple counterpart to bilinearity, but we can combine the statements of **property 3** into one statement as follows:

Let $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{V}$ and $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, and let's expand out $\langle \alpha\mathbf{u} + \beta\mathbf{v}, \gamma\mathbf{w} + \delta\mathbf{z} \rangle$. One use of part (i) gets us to

$$\langle \alpha\mathbf{u}, \gamma\mathbf{w} + \delta\mathbf{z} \rangle + \langle \beta\mathbf{v}, \gamma\mathbf{w} + \delta\mathbf{z} \rangle$$

Then we can use part (ii) twice to get

$$\langle \alpha\mathbf{u}, \gamma\mathbf{w} \rangle + \langle \alpha\mathbf{u}, \delta\mathbf{z} \rangle + \langle \beta\mathbf{v}, \gamma\mathbf{w} \rangle + \langle \beta\mathbf{v}, \delta\mathbf{z} \rangle$$

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1. For all $\mathbf{z} \in \mathbb{V}$, we have that $\langle \mathbf{z}, \mathbf{z} \rangle$ is a non-negative real number, and $\langle \mathbf{z}, \mathbf{z} \rangle = 0$ if and only if $\mathbf{z} = \mathbf{0}$.
2. For all $\mathbf{w}, \mathbf{z} \in \mathbb{V}$, $\langle \mathbf{z}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{z} \rangle}$
3. For all $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{z} \in \mathbb{V}$ and all $\alpha \in \mathbb{C}$ we have

i. $\langle \mathbf{v} + \mathbf{z}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{z}, \mathbf{w} \rangle$
ii. $\langle \mathbf{z}, \mathbf{w} + \mathbf{u} \rangle = \langle \mathbf{z}, \mathbf{w} \rangle + \langle \mathbf{z}, \mathbf{u} \rangle$
iii. $\langle \alpha\mathbf{z}, \mathbf{w} \rangle = \alpha\langle \mathbf{z}, \mathbf{w} \rangle$
iv. $\langle \mathbf{z}, \alpha\mathbf{w} \rangle = \overline{\alpha}\langle \mathbf{z}, \mathbf{w} \rangle$

Now, we can use part (iii) four times to get

$$\alpha\langle \mathbf{u}, \gamma\mathbf{w} \rangle + \alpha\langle \mathbf{u}, \delta\mathbf{z} \rangle + \beta\langle \mathbf{v}, \gamma\mathbf{w} \rangle + \beta\langle \mathbf{v}, \delta\mathbf{z} \rangle$$

And lastly, we use part (iv) four times to get

$$\alpha\overline{\gamma}\langle \mathbf{u}, \mathbf{w} \rangle + \alpha\overline{\delta}\langle \mathbf{u}, \mathbf{z} \rangle + \beta\overline{\gamma}\langle \mathbf{v}, \mathbf{w} \rangle + \beta\overline{\delta}\langle \mathbf{v}, \mathbf{z} \rangle$$

So we see that the inner product is almost bilinear; we simply need to remember to take the conjugate of any scalar we pull out of the right side of the inner product.

Properties of Complex Inner Product Spaces

Example

Show that the standard inner product defined on $\mathbb{C}^n$ is a complex inner product.

Property 1: Let $\vec{z} \in \mathbb{C}^n$. Then

$$\langle \vec{z}, \vec{z} \rangle = \sum_{j=1}^{n} z_j\overline{z_j} = \sum_{j=1}^{n} |z_j|^2$$

Since this is the sum of non-negative real numbers, it must be a non-negative real number. Moreover, the only way to have that

$$\langle \vec{z}, \vec{z} \rangle = \sum_{j=1}^{n} |z_j|^2 = 0$$

is to have $|z_j|^2 = 0$ for all $1 \le j \le n$, and since the only complex number with a modulus of 0 is 0, we see that $\langle \vec{z}, \vec{z} \rangle = 0$ if and only if $\vec{z} = \vec{0}$.

Property 2: Let $\vec{z}, \vec{w} \in \mathbb{C}^n$. Then we can use properties of the conjugate to see that

$$\begin{aligned} \overline{\langle \vec{w}, \vec{z} \rangle} &= \overline{\sum_{j=1}^{n} w_j\overline{z_j}} \\ &= \sum_{j=1}^{n} \overline{w_j}\,\overline{\overline{z_j}} \\ &= \sum_{j=1}^{n} z_j\overline{w_j} \\ &= \langle \vec{z}, \vec{w} \rangle \end{aligned}$$

Properties of Complex Inner Product Spaces

Example

Property 3i: Let $\vec{v}, \vec{w}, \vec{z} \in \mathbb{C}^n$. Then

$$\begin{aligned}\langle \vec{v}+\vec{z}, \vec{w}\rangle &= \sum_{j=1}^{n}(v_j + z_j)\overline{w_j} \\ &= \sum_{j=1}^{n} v_j\overline{w_j} + z_j\overline{w_j} \\ &= \sum_{j=1}^{n} v_j\overline{w_j} + \sum_{j=1}^{n} z_j\overline{w_j} \\ &= \langle \vec{v}, \vec{w}\rangle + \langle \vec{z}, \vec{w}\rangle\end{aligned}$$

Property 3ii: Let $\vec{u}, \vec{w}, \vec{z} \in \mathbb{C}^n$. Then

$$\begin{aligned}\langle \vec{z}, \vec{w}+\vec{u}\rangle &= \sum_{j=1}^{n} z_j\overline{(w_j + u_j)} \\ &= \sum_{j=1}^{n} z_j(\overline{w_j} + \overline{u_j}) \\ &= \sum_{j=1}^{n} z_j\overline{w_j} + z_j\overline{u_j} \\ &= \sum_{j=1}^{n} z_j\overline{w_j} + \sum_{j=1}^{n} z_j\overline{u_j} \\ &= \langle \vec{z}, \vec{w}\rangle + \langle \vec{z}, \vec{u}\rangle\end{aligned}$$

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Example

Property 3iii: Let $\vec{w}, \vec{z} \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}\langle \alpha\vec{z}, \vec{w}\rangle &= \sum_{j=1}^{n} \alpha z_j\overline{w_j} \\ &= \alpha\sum_{j=1}^{n} z_j\overline{w_j} \\ &= \alpha\langle \vec{z}, \vec{w}\rangle\end{aligned}$$

Property 3iv: Let $\vec{w}, \vec{z} \in \mathbb{C}^n$ and $\alpha \in \mathbb{C}$. Then

$$\begin{aligned}\langle \vec{z}, \alpha\vec{w}\rangle &= \sum_{j=1}^{n} z_j\overline{\alpha w_j} \\ &= \sum_{j=1}^{n} z_j(\overline{\alpha}\,\overline{w_j}) \\ &= \sum_{j=1}^{n} z_j(\overline{\alpha})(\overline{w_j}) \\ &= \overline{\alpha}\sum_{j=1}^{n} z_j\overline{w_j} \\ &= \overline{\alpha}\langle \vec{z}, \vec{w}\rangle\end{aligned}$$

Properties of Complex Inner Product Spaces

Theorem 9.5.1

Let $\mathbb{V}$ be a complex inner product space with inner product $\langle\ ,\ \rangle$. Then, for all $\mathbf{w}, \mathbf{z} \in \mathbb{V}$, we have

Cauchy-Schwarz Inequality: $|\langle \mathbf{z}, \mathbf{w} \rangle| \leq \|\mathbf{z}\|\,\|\mathbf{w}\|$

Triangle Inequality: $\|\mathbf{z} + \mathbf{w}\| \leq \|\mathbf{z}\| + \|\mathbf{w}\|$

For once, we cannot simply copy the proof from the proof used in the reals.

Proof (Triangle Inequality)

Note that the Triangle Inequality is equivalent to the statement

$$\|\mathbf{z} + \mathbf{w}\|^2 \leq (\|\mathbf{z}\| + \|\mathbf{w}\|)^2$$

or that

$$\|\mathbf{z} + \mathbf{w}\|^2 - (\|\mathbf{z}\| + \|\mathbf{w}\|)^2 \leq 0$$

Let's expand the left side:

$$\begin{aligned} \|\mathbf{z} + \mathbf{w}\|^2 - (\|\mathbf{z}\| + \|\mathbf{w}\|)^2 &= \|\mathbf{z} + \mathbf{w}\|^2 - (\|\mathbf{z}\|^2 + 2\|\mathbf{z}\|\,\|\mathbf{w}\| + \|\mathbf{w}\|^2) \\ &= \langle \mathbf{z} + \mathbf{w}, \mathbf{z} + \mathbf{w} \rangle - \langle \mathbf{z}, \mathbf{z} \rangle - \langle \mathbf{w}, \mathbf{w} \rangle - 2\|\mathbf{z}\|\,\|\mathbf{w}\| \\ &= \langle \mathbf{z}, \mathbf{z} \rangle + \langle \mathbf{z}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{z} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle - \langle \mathbf{z}, \mathbf{z} \rangle - \langle \mathbf{w}, \mathbf{w} \rangle - 2\|\mathbf{z}\|\,\|\mathbf{w}\| \\ &= \langle \mathbf{z}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{z} \rangle - 2\|\mathbf{z}\|\,\|\mathbf{w}\| \\ &= \langle \mathbf{z}, \mathbf{w} \rangle + \overline{\langle \mathbf{z}, \mathbf{w} \rangle} - 2\|\mathbf{z}\|\,\|\mathbf{w}\| \\ &= 2\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle) - 2\|\mathbf{z}\|\,\|\mathbf{w}\| \end{aligned}$$

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Proof (Triangle Inequality)

So we need to show that

$$2\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle) - 2\|\mathbf{z}\|\,\|\mathbf{w}\| \leq 0$$

which is the same as showing that

$$\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle) \leq \|\mathbf{z}\|\,\|\mathbf{w}\|$$

We will make use of the Cauchy-Schwarz Inequality, by first showing that

$$\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle) \leq |\langle \mathbf{z}, \mathbf{w} \rangle|$$

To see this, we first note that

$$|\langle \mathbf{z}, \mathbf{w} \rangle|^2 = (\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle))^2 + (\mathrm{Im}(\langle \mathbf{z}, \mathbf{w} \rangle))^2$$

Since $(\mathrm{Im}(\langle \mathbf{z}, \mathbf{w} \rangle))^2 \geq 0$, we see that

$$(\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle))^2 \leq |\langle \mathbf{z}, \mathbf{w} \rangle|^2$$

And thus we have that

$$|\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle)| \leq |\langle \mathbf{z}, \mathbf{w} \rangle|$$

But since $\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle) \leq |\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle)|$, we have shown that

$$\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle) \leq |\langle \mathbf{z}, \mathbf{w} \rangle|$$

And the Cauchy-Shwarz Inequality tells us that $|\langle \mathbf{z}, \mathbf{w} \rangle| \leq \|\mathbf{z}\|\,\|\mathbf{w}\|$, so we see that

$$\mathrm{Re}(\langle \mathbf{z}, \mathbf{w} \rangle) \leq |\langle \mathbf{z}, \mathbf{w} \rangle| \leq \|\mathbf{z}\|\,\|\mathbf{w}\|$$

which completes our proof of the Triangle Inequality.