Real Canonical Form

Our eigenvalue $\lambda = a + bi$ for A leads to an eigenvector $\vec{z} = \vec{x} + i\vec{y}$, and we now know that $\text{Span}\{\vec{x}, \vec{y}\}$ is an invariant subspace under A. What next?

We will end up using the real vectors $\vec{x}$ and $\vec{y}$ to form a matrix (instead of the actual eigenvectors, as before), but, to see why, we want to start by looking at the case when A is a 2×2 matrix. Why?

Well, not only is $\text{Span}\{\vec{x}, \vec{y}\}$ a subspace of $\mathbb{R}^n$, but it is specifically a two-dimensional subspace of $\mathbb{R}^n$, with $\mathcal{B} = \{\vec{x}, \vec{y}\}$ as a basis.
In the case when $n = 2$, the only two-dimensional subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$ itself, so we get that the set $\mathcal{B}$ is a basis for $\mathbb{R}^2$.

This means that the matrix $P = \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix}$ can be thought of as a change of coordinates matrix, from standard coordinates to $\mathcal{B}$ coordinates.

And, thus, P is invertible. Moreover, we know that $P^{-1}AP$ will be the matrix for the linear mapping $A\vec{r}$, but with respect to $\mathcal{B}$ coordinates.
To look at this further, let's step back a bit, and define $L : \mathbb{R}^2 \to \mathbb{R}^2$ by $L(\vec{r}) = A\vec{r}$.

So $[L] = A$, and $[L]_\mathcal{B}$ is $\begin{bmatrix} [L(\vec{x})]_\mathcal{B} & [L(\vec{y})]_\mathcal{B} \end{bmatrix} = \begin{bmatrix} [A\vec{x}]_\mathcal{B} & [A\vec{y}]_\mathcal{B} \end{bmatrix}$.

Real Canonical Form

So, we need to find the $\mathcal{B}$ coordinates of $A\vec{x}$ and $A\vec{y}$.

But, we can recall from our work in showing that Span$\mathcal{B}$ is an invariant subspace, that

$$A\vec{x} = a\vec{x} - b\vec{y} \qquad A\vec{y} = b\vec{x} + a\vec{y}$$

Then we see that $[A\vec{x}]_\mathcal{B} = \begin{bmatrix} a \\ -b \end{bmatrix}$ and $[A\vec{y}]_\mathcal{B} = \begin{bmatrix} b \\ a \end{bmatrix}$.

And so we see that $[L]_\mathcal{B} = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

But remember that $[L]_\mathcal{B}$ is the matrix for the linear mapping $A\vec{x}$ with respect to $\mathcal{B}$ coordinates, so we have that $[L]_\mathcal{B} = P^{-1}AP$.

And so we have ended up with a situation similar to diagonalization: we use the eigenvectors to find an invertible matrix P, and $P^{-1}AP$ is a matrix built using the eigenvalues.

Our matrix $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is known as a real canonical form for A.

Real Canonical Form

Definition: Let A be a 2×2 real matrix with eigenvalue $\lambda = a + ib$, $b \neq 0$. The matrix $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ is called a real canonical form for A.

Example

In lecture 3q, we found that $\lambda = 1 + 2i$ is an eigenvalue for $A = \begin{bmatrix} 3 & 4 \\ -2 & -1 \end{bmatrix}$, with corresponding eigenvector $\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i\begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

So, in this case we have $a = 1$, $b = 2$, $\vec{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\vec{y} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$, so we must have that $C = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ is a real canonical form for A, and that $P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$ is such that $P^{-1}AP = C$.

Note: You can easily calculate that $P^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$, and then compute the product $P^{-1}AP$ to verify that it is in fact $\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Real Canonical Form

So, things seem to work quite nicely in the 2×2 case, but we definitely don't have that $\{\vec{x}, \vec{y}\}$ is a basis for $\mathbb{R}^3$.

However, we do know that $\{\vec{v}, \vec{x}, \vec{y}\}$ is a basis for $\mathbb{R}^3$, where $\vec{v}$ is the eigenvector for the real eigenvalue.
For, if A is a 3×3 matrix, then its characteristic polynomial is a degree three polynomial, which must have at least one real root.

In fact, since we know that complex roots of this polynomial will come in conjugate pairs, either A will have three real eigenvalues (counting multiplicity), or one real eigenvalue and two complex eigenvalues (that are conjugates).

Having already looked at the case where A has only real eigenvalues, let's now see we what happens when A has one real eigenvalue μ with eigenvector $\vec{v}$, and complex eigenvalues $a \pm ib$ with eigenvectors $\vec{x} \pm i\vec{y}$.
Theorem 9.4.2 still applies, so we know that $\operatorname{Span}\{\vec{x}, \vec{y}\}$ is a two-dimensional subspace of $\mathbb{R}^3$.

But this is where we make use of the fact that $\operatorname{Span}\{\vec{x}, \vec{y}\}$ does not contain any real eigenvectors, and thus specifically does not contain $\vec{v}$.

So, if we recall the technique for expanding a linearly independent set to a basis, we can start with the linearly independent set $\{\vec{x}, \vec{y}\}$, and add the vector $\vec{v} \notin \operatorname{Span}\{\vec{x}, \vec{y}\}$, and we know that the resulting set $\{\vec{v}, \vec{x}, \vec{y}\}$ is linearly independent.
And since we have a linearly independent set with three vectors, by the two-out-of-three rule, we know that this set is a basis for $\mathbb{R}^3$.

Real Canonical Form

Since $\mathcal{B} = \{\vec{v}, \vec{x}, \vec{y}\}$ is a basis for $\mathbb{R}^3$, the matrix $P = \begin{bmatrix} \vec{v} & \vec{x} & \vec{y} \end{bmatrix}$ is the change of coordinates matrix from standard coordinates to $\mathcal{B}$ coordinates.

And this means that we still have that $P^{-1}AP$ is the matrix for the linear mapping $A\vec{r}$ with respect to $\mathcal{B}$ coordinates.

As we did in the 2×2 case, let's use our knowledge of A to figure out what $P^{-1}AP$ is.

So, let's let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $L(\vec{r}) = A\vec{r}$, so that $[L] = A$. Then

$$\begin{aligned} [L]_{\mathcal{B}} &= \begin{bmatrix} [L(\vec{v})]_{\mathcal{B}} & [L(\vec{x})]_{\mathcal{B}} & [L(\vec{y})]_{\mathcal{B}} \end{bmatrix} \\ &= \begin{bmatrix} [A\vec{v}]_{\mathcal{B}} & [A\vec{x}]_{\mathcal{B}} & [A\vec{y}]_{\mathcal{B}} \end{bmatrix} \end{aligned}$$

Real Canonical Form

So, now we need to find $[A\vec{v}]_{\mathcal{B}}$, $[A\vec{x}]_{\mathcal{B}}$, and $[A\vec{y}]_{\mathcal{B}}$.

We still know that $A\vec{x} = a\vec{x} - b\vec{y}$, so $[A\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ a \\ -b \end{bmatrix}$, and $A\vec{y} = b\vec{x} + a\vec{y}$, so $[A\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ b \\ a \end{bmatrix}$. And since $A\vec{v} = \mu\vec{v}$, we see that $[A\vec{v}]_{\mathcal{B}} = \begin{bmatrix} \mu \\ 0 \\ 0 \end{bmatrix}$.

So we have that

$$P^{-1}AP = \begin{bmatrix} \mu & 0 & 0 \\ 0 & a & b \\ 0 & -b & a \end{bmatrix}$$

And this is a real canonical form for a 3×3 matrix A with one real eigenvalue and two complex eigenvalues.

Real Canonical Form

Example

In lecture 3q, we found that the matrix $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & 1 & 2 \\ -1 & 0 & 3 \end{bmatrix}$ had eigenvalue 1 with corresponding eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and complex eigenvalues $3 \pm i$ with corresponding eigenvectors $\begin{bmatrix} \mp 5i \\ 2 \mp 6i \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix} \pm i \begin{bmatrix} -5 \\ -6 \\ 0 \end{bmatrix}$.

Then we know that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & -1 & 3 \end{bmatrix}$$

is a real canonical form for A, and that the matrix

$$P = \begin{bmatrix} 0 & 0 & -5 \\ 1 & 2 & -6 \\ 0 & 5 & 0 \end{bmatrix}$$

is a change of coordinates matrix that can bring A into this form.