A Review of Vectors in $\mathbb{R}^n$

Definition: $\mathbb{R}^n$ is the set of all vectors of the form $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, where $x_i \in \mathbb{R}$. That is

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \mid x_1, \dots, x_n \in \mathbb{R} \right\}$$

Definition: If $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n$ and $t \in \mathbb{R}$, then we define addition of vectors componentwise by

$$\vec{x} + \vec{y} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

We define scalar multiplication componentwise by

$$t\vec{x} = t\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} tx_1 \\ \vdots \\ tx_n \end{bmatrix}$$

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Theorem 1.2.1

For all $\vec{w}, \vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$ we have

1. $\vec{x} + \vec{y} \in \mathbb{R}^n$ — closed under addition
2. $\vec{x} + \vec{y} = \vec{y} + \vec{x}$ — addition is commutative
3. $(\vec{x} + \vec{y}) + \vec{w} = \vec{x} + (\vec{y} + \vec{w})$ — addition is associative
4. There exists a vector $\vec{0} \in \mathbb{R}^n$ such that $\vec{z} + \vec{0} = \vec{z}$ for all $\vec{z} \in \mathbb{R}^n$ — zero vector
5. For each $\vec{x} \in \mathbb{R}^n$, there exists a vector $-\vec{x} \in \mathbb{R}^n$ such that $\vec{x} + (-\vec{x}) = \vec{0}$. — additive inverse
6. $t\vec{x} \in \mathbb{R}^n$ — closed under scalar multiplication
7. $s(t\vec{x}) = (st)\vec{x}$ — scalar multiplication is associative
8. $(s + t)\vec{x} = s\vec{x} + t\vec{x}$ — vector distribution
9. $t(\vec{x} + \vec{y}) = t\vec{x} + t\vec{y}$ — scalar distribution
10. $1\vec{x} = \vec{x}$ — scalar multiplicative identity

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Definition: If S is the subspace of $\mathbb{R}^n$ consisting of all possible linear combinations of the vectors $\vec{v}_1, \dots, \vec{v}_k \in \mathbb{R}^n$, then S is called a subspace spanned by the set of vectors $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_k\}$, and we say that the set $\mathcal{B}$ spans S. The set $\mathcal{B}$ is called a spanning set for the subspace S. We denote S by

$$S = \text{Span}\{\vec{v}_1, \dots, \vec{v}_k\} = \text{Span}\mathcal{B}$$

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Example

Determine whether or not $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is in $\text{Span}\left\{\begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix}\right\}$.

Solution

We need to determine whether or not there are any solutions to the vector equation $t_1\begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix} + t_2\begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$.

We see that the vector equation has a solution if and only if the following system of linear equations has a solution:

$$\begin{array}{rcrcl} -3t_1 & + & 2t_2 & = & 1 \\ 2t_1 & - & 4t_2 & = & 2 \\ 6t_1 & - & 13t_2 & = & 4 \end{array}$$

We row reduce the corresponding augmented matrix:

$$\left[\begin{array}{cc|c} -3 & 8 & 1 \\ 2 & -4 & 2 \\ 6 & -13 & 4 \end{array}\right] \tfrac{1}{2}R_2 \sim \left[\begin{array}{cc|c} -3 & 8 & 1 \\ 1 & -2 & 1 \\ 6 & -13 & 4 \end{array}\right] R_1 \updownarrow R_2 \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ -3 & 8 & 1 \\ 6 & -13 & 4 \end{array}\right] \begin{array}{l} \\ R_2 + 3R_1 \\ R_3 - 6R_1 \end{array} \sim$$

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 2 & 4 \\ 0 & -1 & -2 \end{array}\right] \tfrac{1}{2}R_2 \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{array}\right] \begin{array}{l} \\ \\ R_3 + R_2 \end{array} \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

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Example

Determine whether or not $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ is in $\text{Span}\left\{\begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix}\right\}$.

Solution

The final matrix, $\left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$, is in row echelon form, and as it has no bad rows, we know that the system is consistent.

This means that $\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ **is** in $\text{Span}\left\{\begin{bmatrix} -3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 8 \\ -4 \\ -13 \end{bmatrix}\right\}$.

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Definition: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is said to be linearly dependent if there exists coefficients $t_1, \dots, t_k$ not all zero such that

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

Definition: A set of vectors $\{\vec{v}_1, \dots, \vec{v}_k\}$ is said to be linearly independent if the only solution to

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

is $t_1 = \cdots = t_k = 0$. This is called the trivial solution.

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Example

Determine whether the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

Solution

We need to see if there are any parameters in the solution of the homogeneous system

$$\begin{array}{rcrcrcl} t_1 & + & t_2 & - & 3t_3 & = & 0 \\ 2t_1 & + & 4t_2 & - & 4t_3 & = & 0 \\ -t_1 & + & 7t_2 & & & = & 0 \end{array}$$

We row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 4 & -4 \\ -1 & 7 & 0 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 2 \\ 0 & 8 & -3 \end{bmatrix} \tfrac{1}{2} R_2 \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 8 & -3 \end{bmatrix} R_3 - 8R_1 \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -11 \end{bmatrix}$$

This last matrix is in REF, and thus we see that the rank of the coefficient matrix is 3.
Since this is the same as the number of variables (which is the same as the number of vectors), there are no parameters in the general solution.
Therefore, the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ **is** linearly independent.

A Review of Vectors in $\mathbb{R}^n$

Example

Determine whether the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is linearly independent.

Solution

We must determine if there are parameters in the solution of the homogeneous system

$$\begin{array}{rcrcrcl} & & t_2 & + & 3t_3 & = & 0 \\ 3t_1 & - & 7t_2 & & & = & 0 \\ -2t_1 & + & 6t_2 & + & 4t_3 & = & 0 \end{array}$$

We row reduce the coefficient matrix:

$$\begin{bmatrix} 0 & 1 & 3 \\ 3 & -7 & 0 \\ -2 & 6 & 4 \end{bmatrix} \tfrac{-1}{2} R_3 \sim \begin{bmatrix} 0 & 1 & 3 \\ 3 & -7 & 0 \\ 1 & -3 & -2 \end{bmatrix} R_1 \updownarrow R_3 \sim \begin{bmatrix} 1 & -3 & -2 \\ 3 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix} R_2 - 3R_1 \sim$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix} R_3 - \tfrac{1}{2} R_2 \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

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Example

Determine whether the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is linearly independent.

Solution

This last matrix, $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$, is in REF, and thus we see that the rank of the coefficient matrix is 2.

This means that there is one parameter in the general solution to the homogeneous system.

Therefore, the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is **not** linearly independent (that is, the set is linearly dependent).