

## Roots of Polynomial Equations

Another use of the complex conjugate is seen in the following theorem.

### Theorem 9.1.2

Let  $p(x) = a_n x^n + \dots + a_1 x + a_0$ , where  $a_j \in \mathbb{R}$  for  $1 \leq j \leq n$ . If  $z$  is a root of  $p(x)$ , then  $\bar{z}$  is also a root of  $p(x)$ .

**Note:** Our polynomial has coefficients from the real numbers, not the complex numbers. This theorem is most certainly not true of polynomials with complex coefficients.

### Proof

Suppose that  $z$  is a root of  $p(x)$ . Then

$$a_n z^n + \dots + a_1 z + a_0 = 0$$

Keeping in mind that our  $a_j$  are also complex numbers, we can use properties of the complex conjugate to see that

$$\begin{aligned} p(\bar{z}) &= a_n \bar{z}^n + \dots + a_1 \bar{z} + a_0 \\ &= \overline{a_n z^n + \dots + a_1 z + a_0} && \text{property 6} \\ &= \overline{0} && \text{property 5} \\ &= \overline{a_n z^n + \dots + a_1 z + a_0} && \text{property 4} \\ &= \bar{0} \\ &= 0 \end{aligned}$$

Thus,  $\bar{z}$  is a root of  $p(x)$ .

## Roots of Polynomial Equations

We can use this fact to help us factor polynomials with real coefficients, and find all their roots.

### Example

Factor the polynomial  $p(x) = x^2 + 9$ , given that  $3i$  is one of the roots.

From our theorem, we know that if  $3i$  is a root, then  $\overline{3i} = -3i$  is also a root.

That means that both  $(x - 3i)$  and  $(x + 3i)$  are factors of  $x^2 + 9$ .

Since  $(x - 3i)(x + 3i) = x^2 + 3ix - 3ix - 9i^2 = x^2 + 9$ , we see that  $x^2 + 9$  factors as  $(x - 3i)(x + 3i)$ .

### Example

Factor  $p(x) = x^3 - 9x^2 + 36x - 54$ , given that  $3 + 3i$  is one of the roots.

From our theorem, we know that if  $3 + 3i$  is one of the roots, then  $\overline{3 + 3i} = 3 - 3i$  is also a root.

This means that both  $(x - 3 - 3i)$  and  $(x - 3 + 3i)$  are factors of  $p$ , so their product is also a factor of  $p$ .

$$(x - 3 - 3i)(x - 3 + 3i) = x^2 - 3x + 3ix - 3x + 9 - 9i - 3ix + 9i - 9i^2 = x^2 - 6x + 18$$

Dividing  $p(x)$  by  $x^2 - 6x + 18$  we get  $(x - 3)$ .

$$p(x) = (x - 3 - 3i)(x - 3 + 3i)(x - 3)$$

### Roots of Polynomial Equations

Could we have started by dividing  $p(x)$  by  $(x - 3 - 3i)$ ?

Yes, and we would have found that  $p(x) = (x - 3 - 3i)(x^2 + (-6 + 3i)x + (9 - 9i))$ .

We can even use the quadratic formula to factor the remaining degree 2 polynomial.

$$\begin{aligned} \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{6 - 3i \pm \sqrt{(-6 + 3i)^2 - 4(1)(9 - 9i)}}{2(1)} \\ &= \frac{6 - 3i \pm \sqrt{27 - 36i - 36 + 36i}}{2} \\ &= \frac{6 - 3i \pm \sqrt{-9}}{2} \\ &= \frac{6 - 3i \pm \sqrt{-1}\sqrt{9}}{2} \\ &= \frac{6 - 3i \pm 3i}{2} \\ &= \frac{6 - 3i + 3i}{2} \quad \text{and} \quad \frac{6 - 3i - 3i}{2} \\ &= 3 \quad \text{and} \quad 3 - 3i \end{aligned}$$

This, of course, confirms our earlier result.

### Roots of Polynomial Equations

#### Example

Factor  $p(x) = x^4 + 4x^3 + 6x^2 + 4x + 5$ , given that  $i$  is one of the roots.

Well, since  $i$  is one of the roots,  $-i$  is also one of the roots, and so we know that both  $(x - i)$  and  $(x + i)$  are factors of this polynomial.

As such, their product  $(x - i)(x + i) = x^2 + ix - ix - i^2 = x^2 + 1$  is a factor of  $p$ . If we divide  $p(x)$  by  $x^2 + 1$ , we get  $x^2 + 4x + 5$ .

Let's plug this into the quadratic formula to find its roots:

$$\begin{aligned} \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2(1)} \\ &= \frac{-4 \pm \sqrt{16 - 20}}{2} \\ &= \frac{-4 \pm \sqrt{-4}}{2} \\ &= \frac{-4 \pm \sqrt{-1}\sqrt{4}}{2} \\ &= \frac{-4 \pm 2i}{2} \\ &= -2 + i \quad \text{and} \quad -2 - i \end{aligned}$$

Note that these two complex roots are conjugates of each other.

So we see that  $p(x) = (x - i)(x + i)(x + 2 - i)(x + 2 + i)$ .