Span and Linear Independence in Polynomials

Definition: Let $\mathcal{B} = \{p_1(x), \ldots, p_k(x)\}$ be a set of polynomials of degree at most n. Then the span of $\mathcal{B}$ is defined as

$$\text{Span}\ \mathcal{B} = \{t_1 p_1(x) + \cdots + t_k p_k(x) \mid t_1, \ldots, t_k \in \mathbb{R}\}$$

Example

Let $\mathcal{B} = \{1 + x + x^2, 1 + 2x + 3x^2, -5 - 5x^2\}$.
Then $-24 + 8x - 20x^2$ is in Span $\mathcal{B}$, since $4(1 + x + x^2) + 2(1 + 2x + 3x^2) + 6(-5 - 5x^2) = -24 + 8x - 20x^2$.

We also have $x^2 \in$ Span $\mathcal{B}$, since $-(1 + x + x^2) + \frac{1}{2}(1 + 2x + 3x^2) - \frac{1}{10}(-5 - 5x^2) = x^2$.

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Example

Let $\mathcal{B} = \{1 + x + x^2, 1 + 2x + 3x^2, -5 - 5x^2\}$.

To see if $6 + x + 6x^2 \in$ Span $\mathcal{B}$, we need to look for a solution to the equation

$$t_1(1 + x + x^2) + t_2(1 + 2x + 3x^2) + t_3(-5 - 5x^2) = 6 + x + 6x^2$$

Performing the calculation on the left, we get

$$\begin{aligned} & t_1(1 + x + x^2) + t_2(1 + 2x + 3x^2) + t_3(-5 - 5x^2) \\ &= (t_1 + t_1 x + t_1 x^2) + (t_2 + 2t_2 x + 3t_2 x^2) + (-5t_3 - 5t_3 x^2) \end{aligned}$$

$$= \begin{array}{rrr} t_1 & +t_1 x & +t_1 x^2 \\ t_2 & +2t_2 x & +3t_2 x^2 \\ -5t_3 & & -5t_3 x^2 \\ \hline (t_1 + t_2 - 5t_3) & +(t_1 + 2t_2)x & +(t_1 + 3t_2 - 5t_3)x^2 \end{array}$$

So we are looking for $t_1, t_2, t_3, \in \mathbb{R}$ such that $(t_1 + t_2 - 5t_3) + (t_1 + 2t_2)x + (t_1 + 3t_2 - 5t_3)x^2 = 6 + x + 6x^2$.

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Example

Let $\mathcal{B} = \{1 + x + x^2, 1 + 2x + 3x^2, -5 - 5x^2\}$.
Setting the coefficients equal to each other, we find that we are looking for a solution to the following system of equations:

$$\begin{array}{rlllll} 1: & t_1 & +t_2 & -5t_3 & = 6 \\ x: & t_1 & +2t_2 & & = 1 \\ x^2: & t_1 & +3t_2 & -5t_3 & = 6 \end{array}$$

To solve this system, we will row reduce the augmented matrix for the system, as follows:

$$\left[\begin{array}{ccc|c} 1 & 1 & -5 & 6 \\ 1 & 2 & 0 & 1 \\ 1 & 3 & -5 & 6 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & -5 & 6 \\ 0 & 1 & 5 & -5 \\ 0 & 2 & 0 & 0 \end{array}\right] R_3 - R_2 \sim \left[\begin{array}{ccc|c} 1 & 1 & -5 & 6 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & -10 & 10 \end{array}\right] - \tfrac{1}{10} R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & -5 & 6 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 1 & -1 \end{array}\right] \begin{array}{l} R_1 + 5R_3 \\ R_2 - 5R_3 \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] R_1 - R_2 \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

So we see that the solution to the system is $t_1 = 1, t_2 = 0, t_3 = -1$.
This means that

$$(1 + x + x^2) + (0)(1 + 2x + 3x^2) - (-5 - 5x^2) = 6 + x + 6x^2$$

and so, yes, $6 + x + 6x^2$ **is** in Span $\mathcal{B}$.

Span and Linear Independence in Polynomials

Definition: The set $\mathcal{B} = \{p_1(x), \ldots, p_k(x)\}$ is said to be linearly independent if the only solution to the equation

$$t_1 p_1(x) + \cdots + t_k p_k(x) = 0$$

is the trivial solution $t_1 = \cdots = t_k = 0$. Otherwise, $\mathcal{B}$ is said to be linearly dependent.

Span and Linear Independence in Polynomials

Example

To determine whether or not $\mathcal{B}$ is linearly independent, where

$$\mathcal{B} = \{1 + 2x + 3x^2 + 4x^3, 2 - x + 7x^2 + 5x^3, -4 - 4x - 8x^2 - 8x^3, -2 - x + 3x^2 + 3x^3\}$$

we need to look for non-trivial solutions to the equation

$$t_1(1 + 2x + 3x^2 + 4x^3) + t_2(2 - x + 7x^2 + 5x^3) + t_3(-4 - 4x - 8x^2 - 8x^3) + t_4(-2 - x + 3x^2 + 3x^3) = 0$$

Performing the calculation on the left, we get

$$\begin{aligned} & t_1(1 + 2x + 3x^2 + 4x^3) + t_2(2 - x + 7x^2 + 5x^3) + t_3(-4 - 4x - 8x^2 - 8x^3) + t_4(-2 - x + 3x^2 + 3x^3) \\ &= (t_1 + 2t_1x + 3t_1x^2 + 4t_1x^3) + (2t_2 - t_2x + 7t_2x^2 + 5t_2x^3) + (-4t_3 - 4t_3x - 8t_3x^2 - 8t_3x^3) + (-2t_4 - t_4x + 3t_4x^2 + 3t_4x^3) \end{aligned}$$

$$= \begin{array}{cccc} t_1 & +2t_1x & +3t_1x^2 & +4t_1x^3 \\ 2t_2 & -t_2x & +7t_2x^2 & +5t_2x^3 \\ -4t_3 & -4t_3x & -8t_3x^2 & -8t_3x^3 \\ -2t_4 & -t_4x & +3t_4x^2 & +3t_4x^3 \\ \hline (t_1 + 2t_2 - 4t_3 - 2t_4) & +(2t_1 - t_2 - 4t_3 - t_4)x & +(3t_1 + 7t_2 - 8t_3 + 3t_4)x^2 & +(4t_1 + 5t_2 - 8t_3 + 3t_4)x^3 \end{array}$$

So, we looking for non-trivial solutions to the equation

$$(t_1 + 2t_2 - 4t_3 - 2t_4) + (2t_1 - t_2 - 4t_3 - t_4)x + (3t_1 + 7t_2 - 8t_3 + 3t_4)x^2 + (4t_1 + 5t_2 - 8t_3 + 3t_4)x^3 = 0 + 0x + 0x^2 + 0x^3$$

Span and Linear Independence in Polynomials

Example

Setting the coefficients equal, we see that this is equivalent to looking for solutions to the following homogeneous system of linear equations:

$$\begin{array}{rrrrl} t_1 & +2t_2 & -4t_3 & -2t_4 & = 0 \\ 2t_1 & -t_2 & -4t_3 & -t_4 & = 0 \\ 3t_1 & +7t_2 & -8t_3 & +3t_4 & = 0 \\ 4t_1 & +5t_2 & -8t_3 & +3t_4 & = 0 \end{array}$$

To solve this system, we row reduce the coefficient matrix as follows:

$$\begin{bmatrix} 1 & 2 & -4 & -2 \\ 2 & -1 & -4 & -1 \\ 3 & 7 & -8 & 3 \\ 4 & 5 & -8 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -4 & -2 \\ 0 & -5 & 4 & 3 \\ 0 & 1 & 4 & 9 \\ 0 & -3 & 8 & 11 \end{bmatrix} R_2 \updownarrow R_3 \sim \begin{bmatrix} 1 & 2 & -4 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & -5 & 4 & 3 \\ 0 & -3 & 8 & 11 \end{bmatrix} \begin{matrix} R_3 + 5R_2 \\ R_4 + 3R_2 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -4 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 24 & 48 \\ 0 & 0 & 20 & 38 \end{bmatrix} (1/24)R_3 \sim \begin{bmatrix} 1 & 2 & -4 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 20 & 38 \end{bmatrix} R_4 - 20R_3 \sim \begin{bmatrix} 1 & 2 & -4 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

The last matrix is in row echelon form, and from it we see that the rank of the coefficient matrix is 4.

Since this is equal to the number of variables, our system has only one solution, specifically the trivial solution, which means that the set $\mathcal{B}$ is linearly independent.