Subspaces

Definition: Suppose that $\mathbb{V}$ is a vector space, and that $\mathbb{U}$ is a subset of $\mathbb{V}$. If $\mathbb{U}$ is a vector space using the same definition of addition and scalar multiplication as $\mathbb{V}$, then $\mathbb{U}$ is called a subspace of $\mathbb{V}$.

Example

Is P_2 a subspace of P_3? Yes!
Since every polynomial of degree up to 2 is also a polynomial of degree up to 3, P_2 is a subset of P_3.
And we already know that P_2 is a vector space, so it is a subspace of P_3.

However, $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$, since the elements of $\mathbb{R}^2$ have exactly two entries, while the elements of $\mathbb{R}^3$ have exactly three entries.
That is to say, $\mathbb{R}^2$ is not a subset of $\mathbb{R}^3$.

Similarly, $M(2,2)$ is not a subspace of $M(2,3)$, because $M(2,2)$ is not a subset of $M(2,3)$.

Subspaces

Why Are Subspaces Important?

A subspace inherits most of the vector space axioms from its parent vector space.

For example, if $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ in $\mathbb{V}$, then this will continue to be true in $\mathbb{U}$, where $\mathbb{U}$ is a subspace of $\mathbb{V}$.
The properties of $\mathbb{V}$ that are inherited by $\mathbb{U}$ are:
V2 Addition is associative.
V5 Addition is commutative.
V7 Scalar multiplication is associative.
V8 Scalar addition is distributive.
V9 Scalar multiplication is distributive.
V10 Scalar multiplicative identity.

In fact, given any subset (but not necessarily a vector space) $\mathbb{W}$ of a vector space $\mathbb{V}$, we know that properties **V2**, **V5**, **V7**, **V8**, **V9**, and **V10** will hold in $\mathbb{W}$.

So, if we want to prove that $\mathbb{W}$ is itself a vector space, we only need to look at properties **V1**, **V3**, **V4**, and **V6**.

Properties **V1** and **V6** were trivial when showing that $\mathbb{R}^n$, $M(m,n)$, and P_n were vector spaces, but this property becomes much more important when we are looking at subspaces.

Subspaces

Example

Let $\mathbb{V} = M(2,2)$, and let $\mathbb{W}$ be the subset of $M(2,2)$ consisting of matrices with at most one non-zero entry.

So, for example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ are all elements of $\mathbb{W}$.

But let's look at the sum $A + B$.

We know that $A + B \in \mathbb{V}$, and even that $A + B = B + A$, since A and B are elements of $\mathbb{V}$.

But it turns out that $A + B \notin \mathbb{W}$, since

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 5 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 0 \end{bmatrix}$$

and thus, $A + B$ has more than one non-zero entry.

Subspaces

When we are considering properties **V1** and **V6**, we are not so much concerned with the existence of $\mathbf{x} + \mathbf{y}$ and $s\mathbf{x}$, as we are that these elements are contained in our smaller set.

Similarly, our concern with **V3** and **V4** is not that $\mathbf{0}$ and $-\mathbf{x}$ exist and satisfy their respective properties, but rather that they are actually elements of $\mathbb{W}$.

But, it turns out that this key fact follows from property **V6** (closure under scalar multiplication).

Well, let $\mathbf{x}$ be any element of $\mathbb{W}$, and suppose that we have already shown that $\mathbb{W}$ is closed under scalar multiplication.
Then we know that $s\mathbf{x} \in \mathbb{W}$ for any $s \in \mathbb{W}$.
But this means that $s\mathbf{x} \in \mathbb{W}$ for $s = 0$ and $s = -1$
And by Theorem 4.2.1, we know that $0\mathbf{x} = \mathbf{0}$ and $(-1)\mathbf{x} = -\mathbf{x}$.
(Again, we get this by using the fact that $\mathbb{V}$ is already known to be a vector space).
So we see that $\mathbf{0} \in \mathbb{W}$ and $-\mathbf{x} \in \mathbb{W}$, as desired.

There is one fine detail I skipped over, though. And that is the fact that I assumed that $\mathbf{x}$ is an element of $\mathbb{W}$.
Well, it only works if $\mathbb{W}$ actually contains elements!
That is, we cannot have $\mathbb{W} = \emptyset$.
Recall that the empty set can never be a vector space, since any vector space must contain at least a zero vector.

Subspaces

Definition: Suppose that $\mathbb{V}$ is a vector space. Then $\mathbb{U}$ is a subspace of $\mathbb{V}$ if is satisfies the following three properties:
S0: $\mathbb{U}$ is a non-empty subset of $\mathbb{V}$
S1: $\mathbf{x}+\mathbf{y} \in \mathbb{U}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{U}$ ($\mathbb{U}$ is closed under addition)
S2: $t\mathbf{x} \in \mathbb{U}$ for all $\mathbf{x} \in \mathbb{U}$ and $t \in \mathbb{R}$ ($\mathbb{U}$ is closed under scalar multiplication)

Subspaces

Example

Show that $\mathbb{U} = \{a + bx + cx^2 \in P_2 \mid a = b = c\}$ is a subspace of P_2.

Solution

First, we check **S0**:

Well, $\mathbb{U}$ is specifically defined as a subset of P_2, and we see that $0 + 0x + 0x^2 \in \mathbb{U}$, so $\mathbb{U}$ is not the empty set.

Next, we check **S1**:

Suppose $p(x), q(x) \in \mathbb{U}$. Then there are $p, q \in \mathbb{R}$ such that $p(x) = p + px + px^2$ and $q(x) = q + qx + qx^2$.
And we have that

$$p(x) + q(x) = (p + px + px^2) + (q + qx + qx^2) = (p + q) + (p + q)x + (p + q)x^2$$

And so we see that $p(x) + q(x) \in \mathbb{U}$.

Finally, we check **S2**:

Suppose $p(x) \in \mathbb{U}$ and $s \in \mathbb{R}$.
Let $p \in \mathbb{R}$ be such that $p(x) = p + px + px^2$.
Then we have that

$$sp(x) = s(p + px + px^2) = (sp) + (sp)x + (sp)x^2$$

And so we see that $sp(x) \in \mathbb{U}$.

And since $\mathbb{U}$ satisfies properties **S0**, **S1**, and **S2**, we have that $\mathbb{U}$ is a subspace of P_2.

Subspaces

Example

Show that $\mathcal{A} = \{a + bx \in P_1 \mid b = a^2\}$ is not a subspace of P_1.

Solution

To show that something is not a subspace, we need to show that any one of the three properties does not hold.

S0 is easy to check so we usually start there.

In this case, $\mathcal{A}$ is obviously a subset of P_1, and we quickly see that the zero polynomial in an element of $\mathcal{A}$, so it is non-empty.

But what about **S1**?

Well, if we had two functions in $\mathcal{A}$, say $a + a^2x$ and $b + b^2x$, then when we add them we get $(a + b) + (a^2 + b^2)x$. For this function to be in $\mathcal{A}$, we need to have that $a^2 + b^2 = (a + b)^2$.
This is, of course, not true in general.

Instead of using a blanket statement of $a^2 + b^2 \neq (a + b)^2$ to show that **S1** fails, the correct course of action is to find specific values of a and b such that $a^2 + b^2 \neq (a + b)^2$, and use them as our counterexample.
One possible choice is $a = 1$ and $b = 2$. Using these values we see that A is not a subspace of P_1:
$\mathcal{A}$ is not a subspace of P_1, since there are elements $1 + x, 2 + 4x \in \mathcal{A}$ such that $(1 + x) + (2 + 4x) = 3 + 5x \notin \mathcal{A}$, so $\mathcal{A}$ is not closed under addition.

The last thing I want to point out is that **S2** also fails to hold in this case.

One counterexample could be that $2 + 4x \in \mathcal{A}$, but $5(2 + 4x) = 10 + 20x \notin \mathcal{A}$.

Subspaces

Example

The set $\mathcal{D}$ of differentiable functions over $\mathbb{R}$ is a vector space.

Solution

We see this by considering it as a subspace of $\mathcal{F}$ (The set of all functions from $\mathbb{R}$ to $\mathbb{R}$). Then we only need to check the three subspace properties:

S0: Differentiable functions are, of course, functions, so $\mathcal{D}$ is a subset of $\mathcal{F}$.
Moveover, $\mathcal{D}$ is not empty, since, for example, the zero function is differentiable.

S1: Suppose f and g are differentiable functions.
Then $f + g$ is also differentiable (in fact, $(f + g)' = f' + g'$), so $f + g \in \mathcal{D}$.

S2: Suppose f is a differentiable function, and $s \in \mathbb{R}$.
Then sf is also differentiable (in fact, $(sf)' = sf'$), so $sf \in \mathcal{D}$.

This same technique tells us that the set C of continuous functions, is a vector space, as is the set $C(a, b)$ of continuous functions on the interval (a, b). Even our polynomial spaces P_n can be thought of as subspaces of $\mathcal{F}$.