The Complex Conjugate

Everything we study comes with its own special operations. For matrices, there's the determinant. For polynomials, there are factoring and evaluating polynomials.
The special operation for complex numbers is called conjugation.

Definition: The complex conjugate of the complex number $z = x + yi$ is $z = x - yi$, and is denoted $\bar{z}$.

Example

$\overline{1+2i} = 1 - 2i$ $\overline{3-4i} = 3 + 4i$ $\bar{5} = 5$

Theorem 9.1.1 (Properties of the Complex Conjugate)

For complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$, we have

1. $\overline{\overline{z_1}} = z_1$
2. z_1 is purely real if and only if $\overline{z_1} = z_1$
3. z_1 is purely imaginary if and only if $\overline{z_1} = -z_1$
4. $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$
5. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$
6. $\overline{z_1^n} = \overline{z_1}^n$
7. $z_1 + \overline{z_1} = 2\text{Re}(z_1) = 2x_1$
8. $z_1 - \overline{z_1} = i2\text{Im}(z_1) = i2y_1$
9. $z_1\overline{z_1} = x_1^2 + y_1^2$

The Complex Conjugate

The proofs of these properties are easy consequences of the definition of conjugation, which makes them great practice! I'll prove properties 1, 2, and 4 now.

Proof of 1

Let $z_1 = x_1 + y_1 i$ be a complex number. Then $\overline{z_1} = \overline{x_1 + y_1 i} = x_1 - y_1 i$. And this means that $\overline{\overline{z_1}} = \overline{x_1 - y_1 i} = x_1 + y_1 i = z_1$, as desired.

Proof of 2

Suppose $z_1 = x_1 + y_1 i$ is a complex number that is purely real. Then $y_1 = 0$ and $z_1 = x_1$. Moreover, we see that $\overline{z_1} = x_1 - y_1 i = x_1 - 0i = x_1 = z_1$. So if z_1 is purely real, then $\overline{z_1} = z_1$.
Now suppose that z_1 is a complex number such that $\overline{z_1} = z_1$. Then $x_1 + y_1 i = x_1 - y_1 i$, so $(x_1 + y_1 i) - (x_1 - y_1 i) = 0$.

This means that $2y_1 i = 0$, which can only happen if $y_1 = 0$. So we see that if $\overline{z_1} = z_1$, then $y_1 = 0$, which means that z_1 is purely real.

The Complex Conjugate

Proof of 4

Let $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ be complex numbers.

$\overline{z_1 + z_2} = \overline{(x_1 + y_1 i) + (x_2 + y_2 i)}$

$= \overline{(x_1 + x_2) + (y_1 + y_2) i}$

$= (x_1 + x_2) - (y_1 + y_2) i$

$= (x_1 + x_2) + (-y_1 - y_2) i$

$= (x_1 - y_1 i) + (x_2 - y_2 i)$

$= \overline{x_1 + y_1 i} + \overline{x_2 + y_2 i}$

$= \overline{z_1} + \overline{z_2}$

We have already noticed that complex conjugation does not affect real numbers.

This will become of great importance during our study of the complex numbers, as it turns out that when we generalize many of the things we did in the real numbers, we will need to introduce complex conjugation to the mix in order to have our formulas turn out correctly.