The Complex Exponential

Repeated uses of the rule for multiplying complex numbers in polar form yields the following theorem.

Theorem 9.1.5 (de Moivre's Formula)

Let $z = r(\cos(\theta) + i\sin(\theta))$, with $r \neq 0$. Then, for any integer n, we have

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Proof

First, we will prove the result for all non-negative integers n by induction. As a base case, note that if $n = 0$, then $z^0 = 1$, and $r^0(\cos(0\theta) + i\sin(0\theta)) = 1(\cos(0) + i\sin(0)) = 1(1 + i0) = 1$, so we have that $z^0 = r^0(\cos(0\theta) + i\sin(0\theta))$.

Now, for our induction step, we assume that $z^k = r^k(\cos(k\theta) + i\sin(k\theta))$ for all complex numbers z. Then we have that

$$\begin{aligned} z^{k+1} &= zz^k \\ &= (r(\cos(\theta) + i\sin(\theta)))(r^k(\cos(k\theta) + i\sin(k\theta))) \\ &= rr^k(\cos(\theta + k\theta) + i\sin(\theta + k\theta)) \\ &= r^{k+1}(\cos((k+1)\theta) + i\sin((k+1)\theta)) \end{aligned}$$

So we have shown that $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ for any non-negative integer n.

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But what if n is a negative integer? Then $m = -n$ is a positive integer, and we have that

$$\begin{aligned} z^n &= z^{-m} \\ &= (z^m)^{-1} \\ &= (r^m(\cos(m\theta) + i\sin(m\theta)))^{-1} \\ &= \left(\frac{1}{r^m}\right)(\cos(-m\theta) + i\sin(-m\theta)) \\ &= r^n(\cos(n\theta) + i\sin(n\theta)) \end{aligned}$$

And so we now have that $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$ for all integers n.
This, by the way, becomes the perfect time to go back and prove property 6 of the complex conjugate: that $\overline{z^n} = (\bar{z})^n$.

Proof

To see this, we want to make use of both de Moivre's formula and Theorem 9.1.a, which says that if $z = r(\cos(\theta) + i\sin(\theta))$, then $\bar{z} = r(\cos(-\theta) + i\sin(-\theta))$. Then we see that

$$\begin{aligned} \overline{z^n} &= \overline{r^n(\cos(n\theta) + i\sin(n\theta))} \\ &= r^n(\cos(-n\theta) + i\sin(-n\theta)) \\ &= (r(\cos(-\theta) + i\sin(-\theta)))^n \\ &= (\bar{z})^n \end{aligned}$$

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Before we pause to look at some examples, I want to push forward a bit more, as we will end up with an easier notation for polar form.
Why didn't we use the easier notation before? Because it comes from the fact that taking the n-th power of a complex number results in multiplying the argument θ by n, not taking its power.
There is another place where we have seen this kind of behavior - when looking at exponentials.
After all, we would get $(x^a)^b = x^{ab}$, not x^{a^b}. So, the function $\cos(\theta) + i\sin(\theta)$ behaves like an exponential function. We use this idea to define the following:

Definition: Euler's Formula says that $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

Definition: For any complex number $z = x + iy$, we define $e^z = e^{x+iy} = e^x e^{iy}$.

We can use Euler's formula to find a new way of writing the polar form of a complex number, since if $z = r(\cos(\theta) + i\sin(\theta))$, then we have $z = re^{i\theta}$. In this form, de Moivre's formula becomes $z^n = r^n e^{in\theta}$.

Example

In the previous lecture we found that $\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$ is a polar form for $1+i$.
We can now write this as $\sqrt{2}e^{i\pi/4}$, and use it to calculate that $(1+i)^4 = (\sqrt{2})^4(e^{(4)(i\pi/4)}) = 4e^{i\pi}$.
If we want, we can even put this back into standard form, by computing that $4e^{i\pi} = 4(\cos(\pi) + i\sin(\pi)) = 4(-1 + i(0)) = -4$.

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Also in the previous lecture, we found that $-3 - \sqrt{3}i = 2\sqrt{3}\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$, so we can now write $-3 - \sqrt{3}i = 2\sqrt{3}e^{i2\pi/3}$.
So we can now compute that $(-3 - \sqrt{3}i)^5 = (2\sqrt{3})^5 e^{(5)(i2\pi)/3} = (12^{1/2})^5 e^{i10\pi/3}$.
We can write $12^{5/2}$ as $288\sqrt{3}$, and we can subtract 2π from $\frac{10\pi}{3}$ to get the equivalent argument $\frac{4\pi}{3}$, to get the final answer of $(-3 - \sqrt{3}i)^5 = 288\sqrt{3}e^{i4\pi/3}$.
And, if we want, we can put our answer back in standard form by computing $288\sqrt{3}e^{i4\pi/3}$

$$\begin{aligned}
&= 288\sqrt{3}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) \\
&= 288\sqrt{3}\left(\left(\frac{-1}{2}\right) + i\left(\frac{-\sqrt{3}}{2}\right)\right) \\
&= -144\sqrt{3} - 432i
\end{aligned}$$

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As one last example, let's find $(1-2i)^{10}$.
In the last lecture, we found that $1-2i \approx \sqrt{5}(\cos(-1.11)+i\sin(-1.11))$.
So we can now say that $1-2i \approx \sqrt{5}e^{(-1.11)i}$, and we get that $(1-2i)^{10} = (5^{1/2})^{10}e^{(10)(-1.11)i} = 3125e^{(-11.1)i}$.
If we want, we can add multiples of 2π (approximately 6.28) to -11.1 to put it in the usual $-\pi$ to 2π range.

Adding two multiples of 2π gives us $(1-2i)^{10} = 3125e^{(1.46)i}$.
In this case I would not recommend putting the answer back into standard form, as the rounding off done during our calculations significantly changes the answer.
(Which is $237+3116i$, by the way. If you are savy with a calculator, you can do the entire calculation, from finding $\theta = \sin^{-1}(-2\sqrt{5})$ to computing $3125\cos(10\theta)$ and $3125\sin(10\theta)$, on your calculator without using any approximations, and you will get this result.)