Vector Spaces

Definition: A vector space over $\mathbb{R}$ is a set $\mathbb{V}$ together with an operation of addition, usually denoted $\mathbf{x}+\mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{V}$, and an operation scalar multiplication, usually denoted $s\mathbf{x}$ for any $\mathbf{x} \in \mathbb{V}$ and $s \in \mathbb{R}$, such that for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{V}$ and $s, t, \in \mathbb{R}$ we have the following properties:

V1. $\mathbf{x}+\mathbf{y} \in \mathbb{V}$ — closed under addition

V2. $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})$ — addition is associative

V3. There is an element $\mathbf{0} \in \mathbb{V}$, (called the zero vector) such that $\mathbf{x}+\mathbf{0} = \mathbf{x} = \mathbf{0}+\mathbf{x}$ — additive identity

V4. For each $\mathbf{x} \in \mathbb{V}$, there exists element $-\mathbf{x}$ such that $\mathbf{x}+(-\mathbf{x}) = \mathbf{0}$ — additive inverse

V5. $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$ — addition is commutative

V6. $s\mathbf{x} \in \mathbb{V}$ — closed under scalar multiplication

V7. $s(t\mathbf{x}) = (st)\mathbf{x}$ — scalar multiplication is associative

V8. $(s+t)\mathbf{x} = s\mathbf{x}+t\mathbf{x}$ — scalar addition is distributive

V9. $s(\mathbf{x}+\mathbf{y}) = s\mathbf{x}+s\mathbf{y}$ — scalar multiplication is distributive

V10. $1\mathbf{x} = \mathbf{x}$ — scalar multiplicative identity

Note: In general, an element of a vector space $\mathbb{V}$ is known as a vector. As elements of $\mathbb{R}^n$ are also known as vectors, this can be confusing, so to help we use the notation $\mathbf{x}$ to mean a vector from a general vector space, and reserve the symbol $\vec{x}$ to mean an element of $\mathbb{R}^n$.

Vector Spaces

Examples

We have already seen that $\mathbb{R}^n$ is a vector space, as well as the set of $m \times n$ matrices, and polynomials of degree up to n.

Notation

1. We write $M(m,n)$ for the vector space of $m \times n$ matrices.
2. We write P_n for the vector space of polynomials of degree up to n.
3. $\mathcal{F}$: the set of all functions from $\mathbb{R}$ to $\mathbb{R}$.
4. $\mathcal{F}(a,b)$: the set of all functions from the interval (a,b) to $\mathbb{R}$.

Vector Spaces

Non-Standard Vector Spaces

Things get interesting when non-standard definitions of addition and scalar multiplication are used.
In these cases, the usual notation for addition and scalar multiplication are replaced with the symbols $\oplus$ and $\odot$.
(Sometimes $\oplus_{\mathbb{V}}$ and $\odot_{\mathbb{V}}$ are used if we need to keep track of which vector space we are referring to.)

Vector Spaces

Non-Standard Vector Spaces

Example

Let $\mathbb{V} = \{(a, b) \mid a, b \in \mathbb{R}, b > 0\}$.
We will define addition in $\mathbb{V}$ by $(a, b) \oplus (c, d) = (ad + bc, bd)$ and we define scalar multiplication in $\mathbb{V}$ by
$t \odot (a, b) = (tab^{t-1}, b^t)$.
Now let's show that $\mathbb{V}$ is a vector space, paying close attention to how the axioms look with our unusual definitions.
To that end, let $(a, b), (c, d), (e, f) \in \mathbb{V}$, and let $s, t \in \mathbb{R}$.

V1. $(a, b) \oplus (c, d) = (ad + bc, bd)$, where $ad + bc \in \mathbb{R}$ and $bd \in \mathbb{R}$, and since both $b > 0$ and $d > 0$, we have that $bd > 0$.
This means that $(ad + bc, bd) \in \mathbb{V}$, and thus $(a, b) \oplus (c, d) \in \mathbb{V}$.

V2. $((a, b) \oplus (c, d)) \oplus (e, f) = (ad + bc, bd) \oplus (e, f) = ((ad + bc)f + (bd)e, (bd)f)$
$= (adf + bcf + bde, bdf) = (a(df) + b(cf + de), b(df))$
$= (a, b) \oplus (cf + de, df) = (a, b) \oplus ((c, d) \oplus (e, f))$.
Note that, thanks to **V1**, we don't need to worry about whether or not any of the intermediate steps are in $\mathbb{V}$.

Vector Spaces

Non-Standard Vector Spaces

Example

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We will define addition in $\mathbb{V}$ by $(a, b) \oplus (c, d) = (ad + bc, bd)$ and we define scalar multiplication in $\mathbb{V}$ by $t \odot (a, b) = (tab^{t-1}, b^t)$.
Now let's show that $\mathbb{V}$ is a vector space, paying close attention to how the axioms look with our unusual definitions.
To that end, let $(a, b), (c, d), (e, f) \in \mathbb{V}$, and let $s, t \in \mathbb{R}$.

V3. To prove this property, we need to find an element $\mathbf{0}$ of $\mathbb{V}$ such that $(a, b) \oplus \mathbf{0} = (a, b)$.
Let's assume that $\mathbf{0} = (x, y)$.
Then we want $(a, b) \oplus (x, y) = (ay - bx, by) = (a, b)$.
So we want $b = by$, which means $y = 1$, and then we want $ay - bx = a$, and plugging in $y = 1$, this becomes $a - bx = a$, so $-bx = 0$.
Now $b \neq 0$, since $b > 0$, so the only way to have $-bx = 0$ is to have $x = 0$.
All this work leads us to the *guess* that $\mathbf{0} = (0, 1)$.
Now we need to prove it.
First we note that $(0, 1) \in \mathbb{V}$, since $0, 1 \in \mathbb{R}$ and $1 > 0$.
Next, we note that $(a, b) \oplus (0, 1) = ((a)(1) + (b)(0), (b)(1)) = (a, b)$ and $(0, 1) \oplus (a, b) = ((0)(b) + (1)(a), (1)(b)) = (a, b)$.
And so we see that property 3 holds, with $(0, 1)$ as our zero vector.

Vector Spaces

Non-Standard Vector Spaces

Example

Let $\mathbb{V} = \{(a, b) \mid a, b \in \mathbb{R}, b > 0\}$.
We will define addition in $\mathbb{V}$ by $(a, b) \oplus (c, d) = (ad + bc, bd)$ and we define scalar multiplication in $\mathbb{V}$ by $t \odot (a, b) = (tab^{t-1}, b^t)$.
Now let's show that $\mathbb{V}$ is a vector space, paying close attention to how the axioms look with our unusual definitions.
To that end, let $(a, b), (c, d), (e, f) \in \mathbb{V}$, and let $s, t \in \mathbb{R}$.

V4. We found in **V3** that $\mathbf{0} = (0, 1)$.
Now, given (a, b), we need to find $-(a, b) \in \mathbb{V}$.
So let's look for (w, z) such that $(a, b) \oplus (w, z) = (0, 1)$.
Well, $(a, b) \oplus (w, z) = (az + bw, bz)$, so we need $bz = 1$ and $az + bw = 0$.
From $bz = 1$ we get that $z = \left(\frac{1}{b}\right)$. (Note that we can divide by b, since $b > 0$.)
Plugging this into $az + bw = 0$, we get $a\left(\frac{1}{b}\right) + bw = 0$, so $w = \left(-\frac{a}{b^2}\right)$.
So we now guess that $-(a, b) = \left(-\frac{a}{b^2}, \frac{1}{b}\right)$.
First, we note that since $b > 0$, $\frac{1}{b} > 0$, so $\left(-\frac{a}{b^2}, \frac{1}{b}\right) \in \mathbb{V}$.
Next, we see that $(a, b) \oplus \left(-\frac{a}{b^2}, \frac{1}{b}\right) = \left(a\left(\frac{1}{b}\right) + b\left(-\frac{a}{b^2}\right), b\left(\frac{1}{b}\right)\right) = \left(\frac{a}{b} - \frac{a}{b}, 1\right) = (0, 1)$, as desired.

Vector Spaces

Non-Standard Vector Spaces

Example

Let $\mathbb{V} = \{(a,b) \mid a, b \in \mathbb{R}, b > 0\}$.
We will define addition in $\mathbb{V}$ by $(a,b) \oplus (c,d) = (ad + bc, bd)$ and we define scalar multiplication in $\mathbb{V}$ by $t \odot (a,b) = (tab^{t-1}, b^t)$.
Now let's show that $\mathbb{V}$ is a vector space, paying close attention to how the axioms look with our unusual definitions.
To that end, let $(a,b), (c,d), (e,f) \in \mathbb{V}$, and let $s, t \in \mathbb{R}$.

V5. $(a,b) \oplus (c,d) = (ad + bc, bd) = (cb + da, db) = (c,d) \oplus (a,b)$

V6. $s \odot (a,b) = (sab^{s-1}, b^s)$.
Since $b > 0$, $b^s > 0$ for any s, and of course $sab^{s-1}, b^s \in \mathbb{R}$, so $(sab^{s-1}, b^s) \in \mathbb{V}$, which means $s \odot (a,b) \in \mathbb{V}$.

V7. $s \odot (t \odot (a,b)) = s \odot ((tab^{t-1}, b^t) = (stab^{t-1}(b^t)^{s-1}, (b^t)^s) = (stab^{t-1}b^{ts-t}, b^{ts})$
$= (stab^{t-1+ts-t}, b^{ts}) = (stab^{ts-1}, b^{ts}) = (stab^{st-1}, b^{st}) = (st) \odot (a,b)$

V8. $(s+t) \odot (a,b) = ((s+t)ab^{s+t-1}, b^{s+t}) = (sab^{s-1}b^t + tab^{t-1}b^s, b^s b^t)$
$= (sab^{s-1}, b^s) \oplus (tab^{t-1}, b^t) = (s \odot (a,b)) \oplus (t \odot (a,b))$

V9. $s \odot ((a,b) \oplus (c,d)) = s \odot (ad + bc, bd) = (s(ad+bc)(bd)^{s-1}, (bd)^s) = (sad(bd)^{s-1} + sbc(bd)^{s-1}, b^s d^s)$
$= (sab^{s-1}d^s + scd^{s-1}b^s, b^s d^s) = (sab^{s-1}, b^s) \oplus (scd^{s-1}, d^s) = (s \odot (a,b)) \oplus (s \odot (c,d))$.

V10. $1 \odot (a,b) = (1ab^{1-1}, b^1) = (ab^0, b) = (a(1), b) = (a,b)$.

Vector Spaces

Theorem 4.2.1

Let $\mathbb{V}$ be a vector space. Then

1. $0\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{V}$
2. $(-1)\mathbf{x} = -\mathbf{x}$ for all $\mathbf{x} \in \mathbb{V}$
3. $t\mathbf{0} = \mathbf{0}$ for all $t \in \mathbb{R}$

Vector Spaces

Proof of $(-1)\mathbf{x} = -\mathbf{x}$ for all $\mathbf{x} \in \mathbb{V}$

Up until now we have been taking it as a notational convention that $(-1)\mathbf{x} = -\mathbf{x}$.
But in this section we introduce the notation $-\mathbf{x}$ to mean the additive inverse of $\mathbf{x}$ and not necessarily the scalar product of -1 times $\mathbf{x}$.
First we prove that the additive inverse is unique.
That is to say, if $\mathbf{x} + \mathbf{y} = \mathbf{0}$ and $\mathbf{x} + \mathbf{z} = \mathbf{0}$, then $\mathbf{y} = \mathbf{z}$.
To see this, let $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ be as stated, and notice that

$$\begin{aligned}
\mathbf{z} &= \mathbf{0} + \mathbf{z} && \text{by the zero vector property} \\
&= (\mathbf{x} + \mathbf{y}) + \mathbf{z} && \text{by our choice of } \mathbf{y} \\
&= (\mathbf{y} + \mathbf{x}) + \mathbf{z} && \text{since addition is commutative} \\
&= \mathbf{y} + (\mathbf{x} + \mathbf{z}) && \text{since addition is associative} \\
&= \mathbf{y} + \mathbf{0} && \text{by our choice of } \mathbf{z} \\
&= \mathbf{y} && \text{by the zero vector property}
\end{aligned}$$

Vector Spaces

Proof of $(-1)\mathbf{x} = -\mathbf{x}$ for all $\mathbf{x} \in \mathbb{V}$

Thanks to the uniqueness of the additive inverse, we now know that in order to show that some $\mathbf{y}$ equals the additive inverse of $\mathbf{x}$ (i.e., to show $\mathbf{y} = -\mathbf{x}$), we need to show that it satisfies the condition in **V4**: $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
In our particular case, we suspect that $-\mathbf{x}$ is $(-1)\mathbf{x}$, and so we will look at $\mathbf{x} + (-1)\mathbf{x}$

$$\begin{aligned}
\mathbf{x} + (-1)\mathbf{x} &= (1)\mathbf{x} + (-1)\mathbf{x} && \text{by the scalar multiplicative identity} \\
&= (1 + (-1))\mathbf{x} && \text{since scalar addition is distributive} \\
&= (0)\mathbf{x} && \text{operation of numbers in } \mathbb{R} \\
&= \mathbf{0} && \text{by Theorem 4.2.1 (1.)}
\end{aligned}$$

And since $\mathbf{x} + (-1)\mathbf{x} = \mathbf{0}$, we know that $(-1)\mathbf{x} = -\mathbf{x}$.

□