Classifying Quadratic Forms

In This Lecture

We will continue our look at quadratic forms. In particular, we will see how to classify quadratic forms and their corresponding symmetric matrices.

Classifying Quadratic Forms

Definition: Let $Q(\vec{x})$ be a quadratic form. We say that $Q(\vec{x})$ is

1. positive definite if $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$.
2. negative definite if $Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$.
3. indefinite if $Q(\vec{x}) > 0$ for some $\vec{x}$ and $Q(\vec{x}) < 0$ for some $\vec{x}$.
4. positive semidefinite if $Q(\vec{x}) \geq 0$ for all $\vec{x}$.
5. negative semidefinite if $Q(\vec{x}) \leq 0$ for all $\vec{x}$.

Example

$Q(x_1, x_2) = x_1^2 + x_2^2$ is positive definite since $Q(x_1, x_2) > 0$ for all $\vec{x} \neq \vec{0}$.

$Q(x_1, x_2) = -x_1^2 - x_2^2$ is negative definite since $Q(x_1, x_2) < 0$ for all $\vec{x} \neq \vec{0}$.

$Q(x_1, x_2) = x_1^2 - x_2^2$ is indefinite since $Q(1, 0) = 1 > 0$ and $Q(0, 1) = -1 < 0$.

$Q(x_1, x_2) = x_1^2$ is positive semidefinite since $Q(x_1, x_2) \geq 0$ for all $\vec{x}$.

Note: $Q(x_1, x_2) = x_1^2$ is equal to 0 for infinitely many values $\vec{x} = (0, x_2)$. So it is not positive definite.

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Note: We classify symmetric matrices by classifying the associated quadratic form.

Example

Classify the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$.

Solution

The corresponding quadratic form is

$$Q(x_1, x_2) = x_1^2 - 2x_2^2$$

We can see that $Q(1,0) = 1$ and $Q(0,1) = -2$. Therefore, $Q(x_1, x_2)$ is indefinite and so A is also indefinite.

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Example

Classify the quadratic form

$$Q(x_1, x_2, x_3, x_4) = 7x_1^2 - 2x_1x_2 + 2x_1x_3 - 6x_1x_4 + 7x_2^2 - 6x_2x_3 + 2x_2x_4 + 7x_3^2 - 2x_3x_4 + 7x_4^2$$

Solution

We will express $Q(\vec{x})$ in diagonal form so that it will be easier to classify.

The symmetric matrix corresponding to $Q(\vec{x})$ is $A = \begin{bmatrix} 7 & -1 & 1 & -3 \\ -1 & 7 & -3 & 1 \\ 1 & -3 & 7 & -1 \\ -3 & 1 & -1 & 7 \end{bmatrix}$.

We can find that the eigenvalues of A are $\lambda_1 = 12$, $\lambda_2 = 8$, $\lambda_3 = 4$, and $\lambda_4 = 4$.

Hence, by Theorem 10.3.1, there exists an orthogonal matrix P such that $\vec{y} = P^T\vec{x}$ gives

$$Q(\vec{x}) = 12y_1^2 + 8y_2^2 + 4y_3^2 + 4y_4^2$$

It is clear that $Q(\vec{x}) > 0$ for all $\vec{y} \neq \vec{0}$, and hence $Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$ (since P is invertible).
Consequently, Q and A are both positive definite.

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Theorem 10.3.2

If A is a symmetric matrix, then A is

1. positive definite if and only if the eigenvalues of A are all positive.
2. negative definite if and only if the eigenvalues of A are all negative.
3. indefinite if and only if some of the eigenvalues of A are positive and some are negative.
4. positive semidefinite if and only if the eigenvalues of A are all non-negative.
5. negative semidefinite if and only if the eigenvalues of A are all non-positive.

The proof is left as an exercise. **Hint:** generalize the method used in the last example.

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Example

Classify the following:
(a) $Q(x_1, x_2) = 4x_1^2 - 6x_1x_2 + 2x_2^2$.

Solution

The corresponding symmetric matrix is $A = \begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix}$.

The characteristic polynomial of A is

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ -3 & 2-\lambda \end{vmatrix} = \lambda^2 - 6\lambda - 1$$

Applying the quadratic formula we get $\lambda = \dfrac{6 \pm \sqrt{40}}{2}$.
So, A has both positive and negative eigenvalues, and hence is indefinite.
Thus, $Q(\vec{x})$ is also indefinite.

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Example

Classify the following:

(b) $Q(x_1, x_2, x_3) = 2x_1^2 + 2x_1x_2 + 2x_1x_3 + 2x_2^2 + 2x_2x_3 + 2x_3^2$.

Solution

The corresponding symmetric matrix is $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

The eigenvalues of A are 1, 1 and 4. Hence, all the eigenvalues of A are positive so A is positive definite. Therefore, $Q(\vec{x})$ is also positive definite.

(c) $A = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 2 & 2 \\ 0 & 2 & 1 \end{bmatrix}$.

Solution

We can find that the eigenvalues of A are 5, 2, and -1.

So, A has both positive and negative eigenvalues. Hence, A is indefinite.