Complex Diagonalization

In this Lecture

- We start extending our theory of diagonalization to the complex case.
- We will see that this does cause some changes.

Complex Diagonalization

Definition: Let $A \in M_{n\times n}(\mathbb{C})$. If there exists $\lambda \in \mathbb{C}$ and $\vec{z} \in \mathbb{C}^n$ with $\vec{z} \neq \vec{0}$ such that $A\vec{z} = \lambda\vec{z}$, then λ is called an eigenvalue of A and $\vec{z}$ is called an eigenvector of A corresponding to λ. We call $(\lambda, \vec{z})$ an eigenpair.

Note: All of the theorems about diagonalization from Linear Algebra 1 still hold, except that we now allow complex eigenvalues and eigenvectors.

Complex Diagonalization

Example

Diagonalize $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ over $\mathbb{C}$.

Solution

The characteristic polynomial is $C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1$.

So, the eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$. The corresponding eigenvectors are $\vec{v}_1 = \begin{bmatrix} i \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ i \end{bmatrix}$.

Thus, taking $P = \begin{bmatrix} i & -1 \\ -1 & i \end{bmatrix}$ gives $P^{-1}AP = D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$.

Notes:

(1) Remember that we are just diagonalizing! We cannot even think about making the eigenvectors unit vectors or orthogonal since we have not yet defined inner products in complex vector spaces.
Do not confuse diagonalization with orthogonal diagonalization.

(2) Although we can diagonalize A, it doesn't mean that it is better for A to be in diagonal form.
In this example, we have taken a simple linear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ and "simplified" it to a linear mapping from $\mathbb{C}^2$ to $\mathbb{C}^2$.

Complex Diagonalization

Example

Diagonalize $A = \begin{bmatrix} 2 & i \\ i & 4 \end{bmatrix}$ over $\mathbb{C}$.

Solution

We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & i \\ i & 4-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda - 3)^2$$

Therefore, the only eigenvalue is $\lambda_1 = 3$ with $a_{\lambda_1} = 2$.

We find that a basis for E_{λ_1} is $\left\{ \begin{bmatrix} i \\ 1 \end{bmatrix} \right\}$.

Therefore, since $g_{\lambda_1} = 1 < 2 = a_{\lambda_1}$, we have that A is not diagonalizable.

Note that our theory for symmetric matrices was for *real* symmetric matrices.

Complex Diagonalization

Example

Diagonalize $A = \begin{bmatrix} 4 & 1+i \\ 1-i & 3 \end{bmatrix}$ over $\mathbb{C}$.

Solution

We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 1+i \\ 1-i & 3-\lambda \end{vmatrix} = \lambda^2 - 7\lambda + 10$$

The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 5$. Therefore, A is diagonalizable.

The corresponding eigenvectors are $\vec{v}_1 = \begin{bmatrix} -1-i \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix}$.

Thus, taking $P = \begin{bmatrix} -1-i & 1+i \\ 2 & 1 \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$.

Complex Diagonalization

Example

Diagonalize $A = \begin{bmatrix} i & 1+i \\ 1-i & 3i \end{bmatrix}$ over $\mathbb{C}$.

Solution

We have

$$C(\lambda) = \det(A - \lambda I) = \begin{vmatrix} i-\lambda & 1+i \\ 1-i & 3i-\lambda \end{vmatrix} = \lambda^2 - 4i\lambda - 5$$

By the quadratic formula, the eigenvalues of A are $\lambda_1 = 1 + 2i$ and $\lambda_2 = -1 + 2i$. Therefore, A is diagonalizable.

The corresponding eigenvectors are $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Thus, taking $P = \begin{bmatrix} 1 & -i \\ 1 & 1 \end{bmatrix}$ gives $P^{-1}AP = \begin{bmatrix} 1+2i & 0 \\ 0 & -1+2i \end{bmatrix}$.

Complex Diagonalization

Theorem 11.3.1

If $A \in M_{n\times n}(\mathbb{R})$ that has a non-real eigenvalue λ with corresponding eigenvector $\vec{z}$, then $\overline{\lambda}$ is also an eigenvalue of A with corresponding eigenvector $\overline{\vec{z}}$.

Proof

We have $A\vec{z} = \lambda\vec{z}$ so taking complex conjugates gives

$$\overline{A\,\vec{z}} = \overline{\lambda\,\vec{z}}$$
$$\overline{A}\,\overline{\vec{z}} = \overline{\lambda}\,\overline{\vec{z}}$$
$$A\,\overline{\vec{z}} = \overline{\lambda}\,\overline{\vec{z}}$$

□

Corollary 11.3.2

If $A \in M_{n\times n}(\mathbb{R})$ and n is odd, then A has at least one real eigenvalue.

Proof

Since A is $n \times n$, its characteristic polynomial $C(\lambda)$ is degree n. Then, by the Fundamental Theorem of Algebra, $C(\lambda)$ has exactly n roots. Since complex roots come in complex conjugate pairs, one root cannot have a pair if n is odd. Thus, $C(\lambda)$ has at least one real root and so A has at least one real eigenvalue.□

Complex Diagonalization

Example

Given that $\lambda_1 = i$ is an eigenvalue of $A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix}$, find the other eigenvalues of A.

Solution

By Theorem 11.3.1, we have that since $\lambda_1 = i$ is an eigenvalue of A, then $\lambda_2 = \overline{\lambda_1} = -i$ is also an eigenvalue of A. The sum of the eigenvalues of a matrix is the trace of the matrix, so the other eigenvalue must satisfy

$$3 = \operatorname{tr} A = \lambda_1 + \lambda_2 + \lambda_3 = i + (-i) + \lambda_3$$

Thus, $\lambda_3 = 3$.