Complex Vector Spaces

In This Lecture

We will extend most of what we did in Linear Algebra 1 with real vector spaces to the complex case.

Complex Vector Spaces

Definition: A set $\mathbb{V}$ is called a vector space over $\mathbb{C}$ or a complex vector space if there is an operation of addition and an operation of scalar multiplication such that for any $\vec{v}, \vec{z}, \vec{w} \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{C}$ we have:

V1 $\vec{z} + \vec{w} \in \mathbb{V}$

V2 $(\vec{z} + \vec{w}) + \vec{v} = \vec{z} + (\vec{w} + \vec{v})$

V3 $\vec{z} + \vec{w} = \vec{w} + \vec{z}$

V4 There exists a vector $\vec{0} \in \mathbb{V}$ such that $\vec{z} + \vec{0} = \vec{z}$ for all $\vec{z} \in \mathbb{V}$

V5 For each $\vec{z} \in \mathbb{V}$ there exists an element $(-\vec{z}) \in \mathbb{V}$ such that $\vec{z} + (-\vec{z}) = \vec{0}$

V6 $\alpha\vec{z} \in \mathbb{V}$

V7 $\alpha(\beta\vec{z}) = (\alpha\beta)\vec{z}$

V8 $(\alpha + \beta)\vec{z} = \alpha\vec{z} + \beta\vec{z}$

V9 $\alpha(\vec{z} + \vec{w}) = \alpha\vec{z} + \alpha\vec{w}$

V10 $1\vec{z} = \vec{z}$

Notes:

- Notice that a vector space over $\mathbb{C}$ has the same definition as a vector space over $\mathbb{R}$ except that we now allow the scalars to be complex numbers.
- All of our concepts and definitions of subspaces, spanning, linear independence, bases, dimension, and coordinates are also exactly the same except that we now allow the use of complex scalars.
- As a result, all of our theory and algorithms are exactly the same.

Complex Vector Spaces

Definition: The set $\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \mid z_i \in \mathbb{C} \right\}$ with addition defined by

$$\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} z_1 + w_1 \\ \vdots \\ z_n + w_n \end{bmatrix}$$

and scalar multiplication defined by

$$\alpha \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} \alpha z_1 \\ \vdots \\ \alpha z_n \end{bmatrix}$$

for any $\alpha \in \mathbb{C}$ is a vector space over $\mathbb{C}$.

Since we allow complex scalar multiplication, we have that the standard basis for $\mathbb{C}^n$ is

$$\{\vec{e}_1, \dots, \vec{e}_n\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\}$$

So, $\dim(\mathbb{C}^n) = n$.

Complex Vector Spaces

Note that just like a complex number $z \in \mathbb{C}$, we can split a vector in $\mathbb{C}^n$ into its real and imaginary parts:

Theorem 11.2.1

If $\vec{z} \in \mathbb{C}^n$, then there exists vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ such that

$$\vec{z} = \vec{x} + i\vec{y}$$

Complex Vector Spaces

Example

Is $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ i \\ 1+i \end{bmatrix}, \begin{bmatrix} 2i \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix} \right\}$ a basis for $\mathbb{C}^3$?

Solution

For linear independence, consider

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ i \\ 1+i \end{bmatrix} + \alpha_2 \begin{bmatrix} 2i \\ -1 \\ 1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ i \end{bmatrix}$$

Calculating the linear combination on the right-hand side and comparing entries gives a homogeneous system of linear equations.

Row reducing the corresponding coefficient matrix gives

$$\begin{bmatrix} 1 & 2i & 1 \\ i & -1 & 1 \\ 1+i & 1 & i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, the system has the unique solution $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so $\mathcal{B}$ is linearly independent.

So we have that $\mathcal{B}$ is a linearly independent set of 3 vectors in the 3-dimensional vector space $\mathbb{C}^3$. Hence, it is a basis for $\mathbb{C}^3$.

Complex Vector Spaces

Example

Find the coordinates of $\vec{z} = \begin{bmatrix} 2i \\ -2+i \\ 3+2i \end{bmatrix}$ with respect to the basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1+i \\ 1-i \end{bmatrix}, \begin{bmatrix} -1 \\ -i \\ -1+2i \end{bmatrix}, \begin{bmatrix} i \\ i \\ 2+2i \end{bmatrix} \right\}$ for $\mathbb{C}^3$.

Solution

We need to find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that

$$\begin{bmatrix} 2i \\ -2+i \\ 3+2i \end{bmatrix} = \alpha_1 \begin{bmatrix} 1 \\ 1+i \\ 1-i \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ -i \\ -1+2i \end{bmatrix} + \alpha_3 \begin{bmatrix} i \\ i \\ 2+2i \end{bmatrix}$$

Calculating the linear combination on the right hand side and comparing entires gives us a complex system of linear equations.

Row reducing gives:

$$\left[\begin{array}{ccc|c} 1 & -1 & i & 2i \\ 1+i & -i & i & -2+i \\ 1-i & -1+2i & 2+2i & 3+2i \end{array}\right] \begin{array}{l} \\ R_2-(1+i)R_1 \\ R_3-(1-i)R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & i & 2i \\ 0 & 1 & 1 & -i \\ 0 & i & 1+i & 1 \end{array}\right] \begin{array}{l} R_1+R_2 \\ \\ R_3-iR_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1+i & i \\ 0 & 1 & 1 & -i \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} R_1-(1+i)R_3 \\ R_2-R_3 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & i \\ 0 & 1 & 0 & -i \\ 0 & 0 & 1 & 0 \end{array}\right] \quad \Rightarrow \quad [\vec{z}]_{\mathcal{B}} = \begin{bmatrix} i \\ -i \\ 0 \end{bmatrix}$$

Complex Vector Spaces

Example

Is $\mathbb{R}^n$ a subspace of $\mathbb{C}^n$?

Solution

Every vector $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ is in $\mathbb{C}^n$ since $x_i \in \mathbb{C}$ for $1 \leq i \leq n$ (each x_i just have an imaginary part of 0).
Thus, $\mathbb{R}^n$ is a non-empty subset of $\mathbb{C}^n$.

For any two vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$, we have that $\vec{x} + \vec{y} \in \mathbb{R}^n$ since we already know that $\mathbb{R}^n$ is closed under addition.

For any scalar t we have that $t\vec{x}$.... **does not have to be in** $\mathbb{R}^n$.
This is the big difference between real and complex vector spaces: we now allow multiplication by complex scalars!

For example, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$, but $(2 + 3i)\vec{e}_1 \notin \mathbb{R}^n$.

So $\mathbb{R}^n$ is not closed under complex scalar multiplication and so is not a vector space over $\mathbb{C}$.
Therefore, $\mathbb{R}^n$ is **not** a subspace of $\mathbb{C}^n$.

Complex Vector Spaces

The other complex vector space that we will refer to in this course is the extension of $M_{m\times n}(\mathbb{R})$.

Definition: The set $M_{m\times n}(\mathbb{C})$ of all $m \times n$ matrices with complex entries is a complex vector space with standard addition and complex scalar multiplication of matrices.

Complex Vector Spaces

We can also extend all of our definitions and theorems about linear mappings (including isomorphisms) to complex vector spaces.

Example

Find the standard matrix of the linear mapping $L : \mathbb{C}^2 \to \mathbb{C}^3$ given by $L(z_1, z_2) = (iz_1, z_2, z_1 + z_2)$.

Solution

The standard basis for $\mathbb{C}^2$ is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

Thus, we have

$$L(1,0) = \begin{bmatrix} i \\ 0 \\ 1 \end{bmatrix}, \qquad L(0,1) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Therefore,

$$[L] = \begin{bmatrix} L(1,0) & L(0,1) \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$$

Complex Vector Spaces

Example

Is the mapping $L : \mathbb{C} \to \mathbb{C}$ defined by $L(z_1) = \overline{z_1}$ a linear mapping?

Solution

For any vectors $\vec{z}_1, \vec{z}_2 \in \mathbb{C}$ and complex scalars $\alpha_1, \alpha_2 \in \mathbb{C}$, consider:

$$L(\alpha_1\vec{z}_1 + \alpha_2\vec{z}_2) = \overline{\alpha_1\vec{z}_1 + \alpha_2\vec{z}_2} = \overline{\alpha_1}\overline{\vec{z}_1} + \overline{\alpha_2}\overline{\vec{z}_2}$$

But $\overline{\alpha} \neq \alpha$ if α is not real. So, we suspect that L is not linear.
To prove it is not linear, we need to provide a counter example:
Taking $\alpha = 1 + i$ and $\vec{z} = 1$, we get

$$L(\alpha\vec{z}) = L[(1+i)1] = \overline{(1+i)(1)} = 1 - i$$

but,

$$\alpha L(\vec{z}) = (1+i)L(1) = 1 + i$$

Since $L(\alpha\vec{z}) \neq \alpha L(\vec{z})$, L is not linear.

Complex Vector Spaces

We also directly extend the concepts of determinants and inverses.

Example

Find the inverse of $Z = \begin{bmatrix} -2i & 1-i \\ 1-3i & 2 \end{bmatrix}$.

Solution

We have $\det Z = (-2i)(2) - (1-i)(1-3i) = 2$.
Thus, from our formula for the inverse of a 2×2 matrix we get

$$Z^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -1+i \\ -1+3i & -2i \end{bmatrix}$$

Complex Vector Spaces

Definition: For any $\vec{z} = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \in \mathbb{C}^n$ and $Z = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & & \vdots \\ z_{m1} & \cdots & z_{mn} \end{bmatrix} \in M_{m\times n}(\mathbb{C})$ we define the complex conjugate by

$$\overline{\vec{z}} = \begin{bmatrix} \overline{z_1} \\ \vdots \\ \overline{z_n} \end{bmatrix} \qquad \overline{Z} = \begin{bmatrix} \overline{z_{11}} & \cdots & \overline{z_{1n}} \\ \vdots & & \vdots \\ \overline{z_{m1}} & \cdots & \overline{z_{mn}} \end{bmatrix}$$

Example

Let $\vec{z} = \begin{bmatrix} 1-3i \\ -3 \\ 2 \\ 4i \end{bmatrix}$, and $Z = \begin{bmatrix} 3 & -2i \\ 1+3i & 1-i \end{bmatrix}$. Then, $\overline{\vec{z}} = \begin{bmatrix} 1+3i \\ -3 \\ 2 \\ -4i \end{bmatrix}$, and $\overline{Z} = \begin{bmatrix} 3 & 2i \\ 1-3i & 1+i \end{bmatrix}$.

Theorem 11.2.2

If $A \in M_{m\times n}(\mathbb{C})$ and $\vec{z} \in \mathbb{C}^n$, then $\overline{A\vec{z}} = \overline{A}\,\overline{\vec{z}}$.