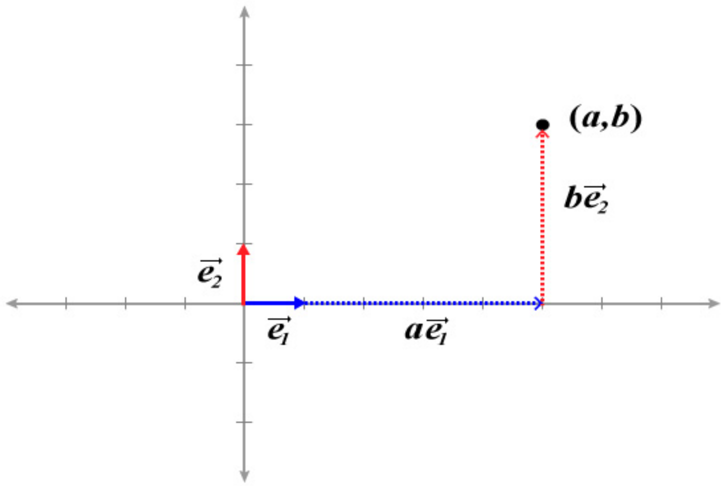
Inner Products



$$\begin{bmatrix} a \\ b \end{bmatrix} = a\vec{e}_1 + b\vec{e}_2$$

Inner Products

Last Lecture

- We saw that every n-dimensional vector space is isomorphic to $\mathbb{R}^n$.
- This means that every n-dimensional vector space must have a nice basis like the standard basis for $\mathbb{R}^n$.

Our goal is to find such nice bases.

What properties does the standard basis have that make it so easy to use?

1. All the vectors in the basis are orthogonal.
2. All the vectors in the basis have unit length.

We have not yet defined the concepts of orthogonality or length in general vector spaces.

In $\mathbb{R}^n$, orthogonality and length are defined in terms of the dot-product.

In This Lecture

We will extend the idea of the dot-product to general vector spaces.

Inner Products

How are we going to define this generalization?
We will use the three important properties of the dot-product to define the general concept called an inner product.

Definition: Let $\mathbb{V}$ be a vector space. An inner product on $\mathbb{V}$ is a function $\langle \, , \rangle : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ such that for all $\vec{v}, \vec{w}, \vec{z} \in \mathbb{V}$ and $a, b \in \mathbb{R}$ we have

1. $\langle \vec{v}, \vec{v} \rangle \geq 0$. Also, $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.
2. $\langle \vec{v}, \vec{w} \rangle = \langle \vec{w}, \vec{v} \rangle$.
3. $\langle a\vec{v} + b\vec{w}, \vec{z} \rangle = a\langle \vec{v}, \vec{z} \rangle + b\langle \vec{w}, \vec{z} \rangle$.

A vector space with an inner product is called an inner product space.

Notes:

- A function with property (1) is said to be positive definite.
- A function with property (2) is called symmetric.
- A function with property (3) is called left linear. Notice that since an inner product is left linear and symmetric, then it is also right linear:
$$\langle \vec{v}, a\vec{w} + b\vec{z} \rangle = a\langle \vec{v}, \vec{w} \rangle + b\langle \vec{v}, \vec{z} \rangle$$
Thus, we say that an inner product is bilinear.
- Just like a vector space is dependent on which definition of addition and scalar multiplication is being used, an inner product space is dependent on which definition of the inner product is being used.

Inner Products

Example

The dot-product on $\mathbb{R}^n$, which we defined in Linear Algebra 1, is an inner product on $\mathbb{R}^n$.
Since it is the most commonly used one, it is called the standard inner product for $\mathbb{R}^n$.

Inner Products

Example

Determine which of the following functions define an inner product on $\mathbb{R}^3$.
(a) $\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3$.

Solution

For (1) we have

$$\langle \vec{x}, \vec{x} \rangle = 2x_1^2 + x_2^2 + 3x_3^2 \geq 0$$

Moreover, the only way this equals 0 is if $x_1 = x_2 = x_3 = 0$. That is, $\langle \vec{x}, \vec{x} \rangle = 0$ if and only if $\vec{x} = \vec{0}$. Thus, this function is positive definite.

For (2) we have

$$\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3 = 2y_1x_1 + y_2x_2 + 3y_3x_3 = \langle \vec{y}, \vec{x} \rangle$$

Hence, it is symmetric.

For (3) we have

$$\begin{aligned} \langle a\vec{x} + b\vec{y}, \vec{z} \rangle &= 2(ax_1 + by_1)z_1 + (ax_2 + by_2)z_2 + 3(ax_3 + by_3)z_3 \\ &= a(2x_1z_1 + x_2z_2 + 3x_3z_3) + b(2y_1z_1 + y_2z_2 + 3y_3z_3) \\ &= a\langle \vec{x}, \vec{z} \rangle + b\langle \vec{y}, \vec{z} \rangle \end{aligned}$$

Hence, the function is bilinear.

Since the function satisfies all three properties of inner products, it is therefore an inner product on $\mathbb{R}^3$.

Inner Products

Example

Determine which of the following functions define an inner product on $\mathbb{R}^3$.
(b) $\langle \vec{x}, \vec{y} \rangle = x_1y_1$.

Solution

This is not an inner product since if $\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then

$$\langle \vec{x}, \vec{x} \rangle = 0(0) = 0$$

But $\vec{x} \neq \vec{0}$, so this function is not positive definite and hence not an inner product.

Inner Products

Example

Determine which of the following functions define an inner product on $\mathbb{R}^3$.

(c) $\langle \vec{x}, \vec{y} \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$.

Solution

We have

$$\left\langle \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\rangle = (2)^2(2)^2 + 0^2(0)^2 + 0^2(0)^2 = 16$$

But,

$$2\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\rangle = 2[(1)^2(2)^2 + 0^2(0)^2 + 0^2(0)^2] = 2[4] = 8$$

Hence,

$$\left\langle \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\rangle = \left\langle 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\rangle \neq 2\left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\rangle$$

Therefore, the function is not bilinear and hence is not an inner product.

Inner Products

Example

Note:

- The inner product $\langle \vec{x}, \vec{y} \rangle = 2x_1y_1 + x_2y_2 + 3x_3y_3$ on $\mathbb{R}^3$ is different than the dot product in $\mathbb{R}^3$.
- In general, a vector space can have infinitely many different inner products.
- However, it can be proven that all the inner products relate in some interesting way. For example, in $\mathbb{R}^n$ all inner products behave like the dot product with respect to some basis.

Inner Products

Example

For all $A, B \in M_{2\times 2}(\mathbb{R})$ we define $\langle A, B\rangle = \operatorname{tr}(B^T A)$. Verify that this is an inner product on $M_{2\times 2}(\mathbb{R})$.

Solution

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$.

For (1) we have

$$\langle A, A\rangle = \operatorname{tr}(A^T A) = \operatorname{tr}\begin{bmatrix} a_1^2 + a_3^2 & ? \\ ? & a_2^2 + a_4^2 \end{bmatrix} = a_1^2 + a_3^2 + a_2^2 + a_4^2$$

This is clearly non-negative and equals 0 if and only if $a_1 = a_2 = a_3 = a_4 = 0$. That is, if A is the zero matrix. Thus, the function is positive definite.

For (2) we have

$$\langle A, B\rangle = \operatorname{tr}(B^T A) = \operatorname{tr}\begin{bmatrix} b_1 a_1 + b_3 a_3 & ? \\ ? & b_2 a_2 + b_4 a_4 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 = \operatorname{tr}(A^T B) = \langle B, A\rangle$$

Therefore, the function is symmetric.

Inner Products

Example

For all $A, B \in M_{2\times 2}(\mathbb{R})$ we define $\langle A, B\rangle = \operatorname{tr}(B^T A)$. Verify that this is an inner product on $M_{2\times 2}(\mathbb{R})$.

Solution

Verifying property (3) is very tedious, so we will not prove that the function is bilinear.
Instead we notice that

$$\langle A, B\rangle = a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

Thus, by its equivalence to the dot product, the function is certainly an inner product. We will call this inner product the standard inner product on $M_{m\times n}(\mathbb{R})$.

Note:
Whenever calculating values of this inner product, we never actually calculate $\operatorname{tr}(B^T A)$, but simply mimic the dot product formula by summing the products of corresponding entries.

Inner Products

Example

Let $A = \begin{bmatrix} 1 & 3 \\ 4 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ be matrices in $M_{2\times2}(\mathbb{R})$.
Calculate $\langle A, B\rangle$ under the standard inner product for $M_{2\times2}(\mathbb{R})$ as defined previously.

Solution

Using our work above, we get

$$\langle A, B\rangle = 1(2) + 3(1) + 4(-1) + (-1)(-1) = 2$$

Note:
Unless we specify otherwise, when we say $\mathbb{R}^n$ or $M_{m\times n}(\mathbb{R})$ we always mean the inner product spaces under the standard inner product.

Inner Products

Example

On $P_2(\mathbb{R})$ the function defined by

$$\langle p(x), q(x)\rangle = p(0)q(0) + p(1)q(1) + p(2)q(2)$$

is an inner product.

The verification is left as an exercise.

Example

Find the inner product of x and $x^2 + x + 1$ using the inner product of $P_2(\mathbb{R})$ defined above.

Solution

We have $p(x) = x$ and $q(x) = x^2 + x + 1$.
Thus,

$$\begin{aligned}\langle p(x), q(x)\rangle &= p(0)q(0) + p(1)q(1) + p(2)q(2) \\ &= 0(1) + 1(3) + 2(7) = 17\end{aligned}$$

Inner Products

Example

On the vector space, $C[a, b]$, of continuous functions defined on $[a, b]$ we define

$$\langle f(x), g(x) \rangle = \int_a^b f(x) g(x) \, dx$$

Example

In $C[-\pi, \pi]$, find $\langle 1, x \rangle$ under the inner product defined above.

Solution

We get

$$\langle 1, x \rangle = \int_{-\pi}^{\pi} 1(x) \, dx = \frac{1}{2} x^2 \Big|_{-\pi}^{\pi} = \frac{1}{2} (\pi)^2 - \frac{1}{2} (-\pi)^2 = 0$$