Isomorphisms

Previously

- Vectors in $\mathbb{R}^n$, as well as matrices, linear mappings, polynomials, and other sets satisfy 10 properties.
- This motivated us to define the abstract concept of a vector space.
- The 10 axioms define a structure for how addition and scalar multiplication work in a vector space.
- All n-dimensional vector spaces should have the same structure.

In This Lecture

- We will develop tools to prove when two vector spaces are the same space.

One-to-One and Onto

Definition: Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces and $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping.

1. If $L(\vec{v}) = L(\vec{u})$ implies $\vec{v} = \vec{u}$, then L is said to be one-to-one (or injective).
2. If for every $\vec{w} \in \mathbb{W}$ there exists a $\vec{v} \in \mathbb{V}$ such that $L(\vec{v}) = \vec{w}$, then L is said to be onto (or surjective).

One-to-One and Onto

Lemma 8.4.1

Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. L is one-to-one if and only if $\text{Ker}(L) = \{\vec{0}\}$.

Proof

Assume that L is one-to-one.
Let $\vec{x} \in \text{Ker}(L)$. Then, $L(\vec{x}) = \vec{0}$.
But, $L(\vec{0}) = \vec{0}$. Hence, we have that $L(\vec{x}) = L(\vec{0})$.
Since L is one-to-one, this implies that $\vec{x} = \vec{0}$. Therefore, $\text{Ker}(L) = \{\vec{0}\}$.

Assume that $\text{Ker}(L) = \{\vec{0}\}$.
We consider $L(\vec{u}) = L(\vec{v})$.
Then

$$\begin{aligned} L(\vec{u}) - L(\vec{v}) &= \vec{0} \\ L(\vec{u} - \vec{v}) &= \vec{0} \end{aligned}$$

This implies that $\vec{u} - \vec{v} \in \text{Ker}(L)$.
Hence, $\vec{u} - \vec{v} = \vec{0}$. So, $\vec{u} = \vec{v}$.

Thus, L is one-to-one.

□

Isomorphisms

For two vector spaces $\mathbb{V}$ and $\mathbb{W}$ to be the 'same' we want for each vector $\vec{v} \in \mathbb{V}$ there to be a unique corresponding vector in $\mathbb{W}$, and linear combinations of corresponding vectors to result in corresponding vectors.

If for each $\vec{v} \in \mathbb{V}$ there is a unique corresponding vector in $\mathbb{W}$, then there should be a one-to-one and onto mapping $L : \mathbb{V} \to \mathbb{W}$. Moreover, if linear combinations are going to be preserved, then L must be linear.

Definition: Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. We say that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if there exists a linear mapping $L : \mathbb{V} \to \mathbb{W}$ that is one-to-one and onto. Such a mapping L is called an isomorphism.

Note: Two vector spaces being isomorphic means that the two vector spaces really are the same vector space.

Example

Observe that

$$(1 + 2x + 3x^2 + 4x^3) + (5 + 6x + 7x^2 + 8x^3) = 6 + 8x + 10x^2 + 12x^3$$

and

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

Isomorphisms

Example

Prove that $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$ are isomorphic.

Solution

Let $L : P_3(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ be defined by $L(a+bx+cx^2+dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Linear: Left as an exercise.

One-To-One: Observe that

$$L(a+bx+cx^2+dx^3) = L(e+fx+gx^2+hx^3) \Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

Hence, $a=e$, $b=f$, $c=g$, and $d=h$. Thus, $a+bx+cx^2+dx^3 = e+fx+gx^2+hx^3$, so L is one-to-one.

Onto:

Pick $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{2\times2}(\mathbb{R})$. We see that $L(a+bx+cx^2+dx^3) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so L is onto.

Hence, L is an isomorphism and so $P_3(\mathbb{R})$ and $M_{2\times2}(\mathbb{R})$ are isomorphic.

Note: If $\mathbb{V}$ and $\mathbb{W}$ are isomorphic, then it does not mean every linear mapping $L : \mathbb{V} \to \mathbb{W}$ must be an isomorphism. For example, $T : P_3(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T(a+bx+cx^2+dx^3) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is definitely not one-to-one nor onto!

Isomorphisms

Example

Prove that $P_2(\mathbb{R})$ and $\mathbb{V} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \mid x_1+x_2+x_3+x_4 = 0 \right\}$ are isomorphic.

Solution

Observe that $\{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$ and $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $\mathbb{V}$.

Thus, we define $L : P_2(\mathbb{R}) \to \mathbb{V}$ by

$$L(a+bx+cx^2) = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ -a-b-c \end{bmatrix}$$

Isomorphisms

Example

Prove that $P_2(\mathbb{R})$ and $\mathbb{V} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0 \right\}$ are isomorphic.

Solution

Linear: Left as an exercise.

One-to-One:

Let $a + bx + cx^2 \in \text{Ker}(L)$. Then, $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = L(a + bx + cx^2) = \begin{bmatrix} a \\ b \\ c \\ -a-b-c \end{bmatrix}$

Hence, $a = 0$, $b = 0$, and $c = 0$. Thus, $a + bx + cx^2 = 0$, so $\text{Ker}(L) = \{0\}$ and so L is one-to-one by Lemma 8.4.1.

Onto: Pick any vector $\vec{v} \in \mathbb{V}$. Since $\vec{v} \in \mathbb{V}$ we can write it as a linear combination of the basis vectors. Say,

$$\vec{v} = a\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

Hence, we have that $L(a + bx + cx^2) = \vec{v}$ and so L is also onto.

Therefore, L is an isomorphism. So, $P_2(\mathbb{R})$ and $\mathbb{V}$ are isomorphic.