## Isomorphisms

### Last Lecture

- We conjectured that for two vector spaces to be isomorphic, they must have the same dimension.
- We conjectured that an isomorphism must map basis vectors to basis vectors.

### In This Lecture

- We will prove both of these conjectures, and more!

## Isomorphisms

**Theorem 8.4.2**

If $\mathbb{V}$ and $\mathbb{W}$ are finite dimensional vector spaces, then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if they have the same dimension.

### Proof

Assume that $\mathbb{V}$ and $\mathbb{W}$ are isomorphic and $\dim \mathbb{V} = n$.
Since $\mathbb{V}$ and $\mathbb{W}$ are isomorphic, there exists an isomorphism $L : \mathbb{V} \to \mathbb{W}$. If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{V}$, then we want to prove that $\mathcal{C} = \{L(\vec{v}_1), \dots, L(\vec{v}_n)\}$ is a basis for $\mathbb{W}$.
Consider

$$\begin{aligned} c_1L(\vec{v}_1) + \cdots + c_nL(\vec{v}_n) &= \vec{0}_{\mathbb{W}} \\ L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) &= \vec{0}_{\mathbb{W}} \end{aligned}$$

Therefore, $c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in \text{Ker}(L)$.
But, we also know that $L$ is one-to-one, and hence $\text{Ker}(L) = \{\vec{0}_{\mathbb{V}}\}$ by Lemma 8.4.1.
Thus,

$$c_1\vec{v}_1 + \cdots + c_n\vec{v}_n = \vec{0}_{\mathbb{V}}$$

which implies that $c_1 = \cdots = c_n = 0$ since $\mathcal{B}$ is linearly independent.
Thus, $\mathcal{C}$ is linearly independent.

## Isomorphisms

**Theorem 8.4.2**

If $\mathbb{V}$ and $\mathbb{W}$ are finite dimensional vector spaces, then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if they have the same dimension.

### Proof

Let $\vec{w} \in \mathbb{W}$. Because $L$ is onto, there exists $\vec{v} \in \mathbb{V}$ such that $L(\vec{v}) = \vec{w}$.
Since $\mathcal{B}$ is a basis for $\mathbb{V}$ we can write

$$\begin{aligned} \vec{v} &= d_1\vec{v}_1 + \cdots + d_n\vec{v}_n \\ \vec{w} &= L(d_1\vec{v}_1 + \cdots + d_n\vec{v}_n) \\ &= d_1L(\vec{v}_1) + \cdots + d_nL(\vec{v}_n) \end{aligned}$$

So, $\mathcal{C}$ also spans $\mathbb{W}$ and hence $\mathcal{C}$ is a basis for $\mathbb{W}$.
Consequently,

$$\dim \mathbb{W} = n = \dim \mathbb{V}$$

## Isomorphisms

**Theorem 8.4.2**

If $\mathbb{V}$ and $\mathbb{W}$ are finite dimensional vector spaces, then $\mathbb{V}$ and $\mathbb{W}$ are isomorphic if and only if they have the same dimension.

### Proof

Assume $\dim \mathbb{V} = n = \dim \mathbb{W}$.
Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a basis for $\mathbb{V}$, and let $\mathcal{C} = \{\vec{w}_1, \ldots, \vec{w}_n\}$ be a basis for $\mathbb{W}$.
We define $L : \mathbb{V} \to \mathbb{W}$ by

$$L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n$$

*Linear:* Left as an exercise.

*One-To-One:* Assume that $\vec{x} = c_1\vec{v}_1 + \cdots + c_n\vec{v}_n \in \operatorname{Ker}(L)$. Then,

$$\vec{0} = L(\vec{x}) = L(c_1\vec{v}_1 + \cdots + c_n\vec{v}_n) = c_1\vec{w}_1 + \cdots + c_n\vec{w}_n$$

Since $\mathcal{C}$ is linearly independent we get that $c_1 = \cdots = c_n = 0$. This implies that $\vec{x} = \vec{0}$, so $\operatorname{Ker}(L) = \{\vec{0}_{\mathbb{V}}\}$, so $L$ is one-to-one by Lemma 8.4.1.

*Onto:* Since $\operatorname{Ker}(L) = \{\vec{0}_{\mathbb{V}}\}$ by the Rank-Nullity Theorem we get that

$$\operatorname{rank}(L) = \dim \mathbb{V} - \operatorname{nullity}(L) = \dim \mathbb{V} - 0 = \dim \mathbb{V} = \dim \mathbb{W}$$

Thus, $\operatorname{Range}(L) = \mathbb{W}$, and hence $L$ is also onto.

Therefore, $L$ is an isomorphism and so $\mathbb{V}$ and $\mathbb{W}$ are isomorphic.

□

## Isomorphisms

**Theorem 8.4.3**

If $\mathbb{V}$ and $\mathbb{W}$ are $n$-dimensional vector spaces, and $L : \mathbb{V} \to \mathbb{W}$ is linear, then $L$ is one-to-one if and only if $L$ is onto.

**Note:** Again, it is very important to remember that just because two vector spaces $\mathbb{V}$ and $\mathbb{W}$ are isomorphic, it does not mean that every linear mapping $L : \mathbb{V} \to \mathbb{W}$ is an isomorphism.

One nice result that is easy to prove using isomorphism is:

**Theorem 8.4.4**

If $\mathcal{B}$ is any basis for an $n$-dimensional vector space $\mathbb{V}$, $\mathcal{C}$ is any basis for an $m$-dimensional vector space $\mathbb{W}$, and $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then

$$\operatorname{rank}(L) = \operatorname{rank}({}_{\mathcal{C}}[L]_{\mathcal{B}})$$

The proof is left as an exercise.