Linear Mappings

Previously

- We looked at the four fundamental subspaces of a matrix.
- We saw that linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ are connected to matrices through the matrix of a linear mapping.

In This Lecture

- We will review the connection between columnspace and nullspace of the standard matrix of a linear mapping to the range and kernel of the linear mapping.
- We will generalize the definition of a linear mapping.

Linear Mappings

Definition: A mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is said to be linear if

$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$.

Definition: The range of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\text{Range}(L) = \{L(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\}$$

Definition: The kernel of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\text{Ker}(L) = \{\vec{x} \in \mathbb{R}^n \mid L(\vec{x}) = \vec{0}\}$$

Definition: The standard matrix of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$[L] = \begin{bmatrix} L(\vec{e}_1) & \cdots & L(\vec{e}_n) \end{bmatrix}$$

It satisfies

$$L(\vec{x}) = [L]\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$.

Linear Mappings

Theorem 7.2.1

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $\mathrm{Range}(L) = \mathrm{Col}([L])$.

Proof

We have

$$\begin{aligned}\mathrm{Range}(L) &= \{L(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\} \\ &= \{[L]\vec{x} \mid \vec{x} \in \mathbb{R}^n\} \\ &= \mathrm{Col}([L])\end{aligned}$$

We get a similar result for the kernel of L. □

Theorem 7.2.2

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $\mathrm{Ker}(L) = \mathrm{Null}([L])$.

The proof is left as an exercise.

Linear Mappings

Theorem 7.2.3

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then,

$$\dim\big(\mathrm{Range}(L)\big) + \dim(\mathrm{Ker}(L)) = \dim(\mathbb{R}^n)$$

We will soon see that the generalization of this theorem is very useful.

General Linear Mappings

We will now look at linear mappings whose domain is a vector space $\mathbb{V}$ and whose codomain is a vector space $\mathbb{W}$.

Note: It is important not to assume that any results that held for linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ also hold for linear mappings $L : \mathbb{V} \to \mathbb{W}$. Our goal will be to prove which results are the same and which are not.

Definition: Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. A mapping $L : \mathbb{V} \to \mathbb{W}$ is called a linear mapping if

$$L(t\vec{x} + s\vec{y}) = tL(\vec{x}) + sL(\vec{y})$$

for all $\vec{x}, \vec{y} \in \mathbb{V}$ and $s, t \in \mathbb{R}$.

General Linear Mappings

Example

Let $L : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b+c)x + (a-b-d)x^2$.

(a) Evaluate $L\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\right)$.

Solution

$$L\left(\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\right) = [1 + 2 + (-1)]x + (1 - 2 - 1)x^2 = 2x - 2x^2$$

General Linear Mappings

Example

Let $L : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b+c)x + (a-b-d)x^2$.

(b) Find a matrix A such that $L(A) = 2x + x^2$.

Solution

We need to find $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$2x + x^2 = L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b+c)x + (a-b-d)x^2$$

Hence, we need $a+b+c = 2$ and $a-b-d = 1$.
Solving this system, we see that one choice is $a = 3/2$, $b = 1/2$, $c = 0$, and $d = 0$.
That is,

$$L\left(\begin{bmatrix} 3/2 & 1/2 \\ 0 & 0 \end{bmatrix}\right) = 2x + x^2$$

General Linear Mappings

Example

Let $L : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ be defined by $L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b+c)x + (a-b-d)x^2$.

(c) Prove that L is linear.

Solution

Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_{2\times 2}(\mathbb{R})$ and $s, t \in \mathbb{R}$.

$$\begin{aligned} L\left(s\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + t\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} sa_1 + ta_2 & sb_1 + tb_2 \\ sc_1 + tc_2 & sd_1 + td_2 \end{bmatrix}\right) \\ &= ([sa_1 + ta_2] + [sb_1 + tb_2] + [sc_1 + tc_2])x + ([sa_1 + ta_2] - [sb_1 + tb_2] - [sd_1 + td_2])x^2 \\ &= s[(a_1 + b_1 + c_1)x + (a_1 - b_1 - d_1)x^2] + t[(a_2 + b_2 + c_2)x + (a_2 - b_2 - d_2)x^2] \\ &= sL\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + tL\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \end{aligned}$$

Hence, L is linear.

General Linear Mappings

Theorem 8.1.1

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces and let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Then,

$$L(\vec{0}_{\mathbb{V}}) = \vec{0}_{\mathbb{W}}$$

The proof is left as an exercise.

Operations on Linear Mappings

Definition: Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces and let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ be linear mappings. We define $L + M : \mathbb{V} \to \mathbb{W}$ by

$$(L + M)(\vec{v}) = L(\vec{v}) + M(\vec{v}), \qquad \text{for all } \vec{v} \in \mathbb{V}$$

For any $t \in \mathbb{R}$ we define $tL : \mathbb{V} \to \mathbb{W}$ by

$$(tL)(\vec{v}) = tL(\vec{v}), \qquad \text{for all } \vec{v} \in \mathbb{V}$$

Notes:

- We have defined $L + M$ and tL for every vector $\vec{v} \in \mathbb{V}$, so the domain of these mappings is $\mathbb{V}$.
- Since $L(\vec{v})$ and $M(\vec{v})$ are in $\mathbb{W}$ and $t \in \mathbb{R}$, then $L(\vec{v}) + M(\vec{v})$ and $tL(\vec{v})$ are in $\mathbb{W}$ since $\mathbb{W}$ is closed under addition and scalar multiplication since it is a vector space. Thus, the codomain of $L + M$ and tL is $\mathbb{W}$.

Operations on Linear Mappings

Theorem 8.1.2

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. The set $\mathbb{L}$ of all linear mappings $L : \mathbb{V} \to \mathbb{W}$ is a vector space.

Proof

Let L, M be linear mappings in the set $\mathbb{L}$.

[V1] To prove that $\mathbb{L}$ is closed under addition, we need to show that $L + M$ is a linear mapping.
For all $\vec{v}_1, \vec{v}_2 \in \mathbb{V}$ and $s, t \in \mathbb{R}$ we have

$$\begin{aligned}(L+M)(s\vec{v}_1 + t\vec{v}_2) &= L(s\vec{v}_1 + t\vec{v}_2) + M(s\vec{v}_1 + t\vec{v}_2)\\ &= sL(\vec{v}_1) + tL(\vec{v}_2) + sM(\vec{v}_1) + tM(\vec{v}_2)\\ &= s[L(\vec{v}_1) + M(\vec{v}_1)] + t[L(\vec{v}_2) + M(\vec{v}_2)]\\ &= s(L+M)(\vec{v}_1) + t(L+M)(\vec{v}_2)\end{aligned}$$

Thus, $L + M \in \mathbb{L}$.

[V2] For any $\vec{v} \in \mathbb{V}$ we have

$$\begin{aligned}(L+M)(\vec{v}) &= L(\vec{v}) + M(\vec{v})\\ &= M(\vec{v}) + L(\vec{v}) \text{ since addition in } \mathbb{W} \text{ is commutative}\\ &= (M+L)(\vec{v})\end{aligned}$$

□

Operations on Linear Mappings

Definition: Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ be linear mappings. We define $M \circ L : \mathbb{V} \to \mathbb{U}$ by $(M \circ L)(\vec{v}) = M(L(\vec{v}))$ for any $\vec{v} \in \mathbb{V}$.

Theorem 8.1.3

If $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ are linear mappings, then $M \circ L$ is a linear mapping from $\mathbb{V}$ to $\mathbb{U}$.