Matrix Mappings

In This Lecture

- We will continue our examination of general linear mappings $L : \mathbb{V} \to \mathbb{W}$.
- We will notice some differences between our previous results and this general case.

Matrix Mappings

We now show that every linear mapping $L : \mathbb{V} \to \mathbb{W}$ can also be represented as a matrix mapping.
However, we must be careful when dealing with general vector spaces as our domain and codomain.

For example, it is impossible to represent a linear mapping $L : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ as a matrix mapping of the form $L(\vec{x}) = A\vec{x}$ since we can not multiply a matrix A by a polynomial in $P_2(\mathbb{R})$. Moreover, we would require the result to be a 2×2 matrix.

To make this work, we must find a way to represent any vector in any vector space as a vector in $\mathbb{R}^n$. To do this, we will use coordinates.

Definition: If $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for a vector space $\mathbb{V}$ and $\vec{v} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n \in \mathbb{V}$, then the coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$ is

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

We will use coordinates of a vector to turn polynomials in $P_2(\mathbb{R})$ into a vector in $\mathbb{R}^3$.
We can interpret $A[\vec{x}]_{\mathcal{B}}$ as the coordinate vector of the image with respect to some basis for $M_{2\times 2}(\mathbb{R})$.

$$[L(\vec{x})]_{\mathcal{C}} = A[\vec{x}]_{\mathcal{B}}$$

where $\mathcal{B}$ is a basis for $P_2(\mathbb{R})$ and $\mathcal{C}$ is a basis for $M_{2\times 2}(\mathbb{R})$.

Matrix Mappings

Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping, let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{V}$ and $\mathcal{C}$ be a basis for $\mathbb{W}$. For any $\vec{v} \in \mathbb{V}$ we want to define a matrix A such that

$$[L(\vec{v})]_{\mathcal{C}} = A[\vec{v}]_{\mathcal{B}} \qquad \text{for all } \vec{v} \in \mathbb{V}$$

Consider the left-hand side $[L(\vec{v})]_{\mathcal{C}}$.
Using properties of linear mappings and coordinates, we get

$$\begin{aligned} [L(\vec{v})]_{\mathcal{C}} &= [L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n)]_{\mathcal{C}} \\ &= [b_1 L(\vec{v}_1) + \cdots + b_n L(\vec{v}_n)]_{\mathcal{C}} \\ &= b_1[L(\vec{v}_1)]_{\mathcal{C}} + \cdots + b_n[L(\vec{v}_n)]_{\mathcal{C}} \\ &= \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \cdots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \\ &= A[\vec{v}]_{\mathcal{B}} \end{aligned}$$

Thus, we see the desired matrix is

$$A = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \cdots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

Matrix of a Linear mapping

Definition: Suppose $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is any basis for a vector space $\mathbb{V}$ and $\mathcal{C}$ is any basis for a finite dimensional vector space $\mathbb{W}$. Then the matrix of $L : \mathbb{V} \to \mathbb{W}$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$ is

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \cdots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

It satisfies

$$[L(\vec{v})]_{\mathcal{C}} = {}_{\mathcal{C}}[L]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}}, \qquad \text{for all } \vec{v} \in \mathbb{V}$$

Note the following:

- The forward subscript of the matrix of the linear mapping is the basis of the domain and the backward subscript is the basis of the codmain.
- If $\mathbb{V} = \mathbb{R}^n$ and $\mathbb{W} = \mathbb{R}^m$ and $\mathcal{B}$ and $\mathcal{C}$ are the respective standard bases, then this matches the definition of the standard matrix.

Matrix of a Linear mapping

Example

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for $\mathbb{V}$ and $\mathcal{C} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$ be a basis for $\mathbb{W}$. If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping such that

$$L(\vec{v}_1) = 2\vec{w}_1 + 3\vec{w}_2 - \vec{w}_4$$
$$L(\vec{v}_2) = \vec{w}_1 + 3\vec{w}_2 + 2\vec{w}_3 - \vec{w}_4$$
$$L(\vec{v}_3) = -\vec{w}_1 + 2\vec{w}_4$$

find the matrix ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ of L and use it to find $L(\vec{x})$ where $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

Solution

By definition, we have

$$ {}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & [L(\vec{v}_2)]_{\mathcal{C}} & [L(\vec{v}_3)]_{\mathcal{C}} \end{bmatrix} $$

We have

$$[L(\vec{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ -1 \end{bmatrix} \qquad [L(\vec{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ -1 \end{bmatrix} \qquad [L(\vec{v}_3)]_{\mathcal{C}} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

Hence,

$$ {}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & [L(\vec{v}_2)]_{\mathcal{C}} & [L(\vec{v}_3)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 3 & 0 \\ 0 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

Matrix of a Linear mapping

Example

Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ be a basis for $\mathbb{V}$ and $\mathcal{C} = \{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$ be a basis for $\mathbb{W}$. If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping such that

$$L(\vec{v}_1) = 2\vec{w}_1 + 3\vec{w}_2 - \vec{w}_4$$
$$L(\vec{v}_2) = \vec{w}_1 + 3\vec{w}_2 + 2\vec{w}_3 - \vec{w}_4$$
$$L(\vec{v}_3) = -\vec{w}_1 + 2\vec{w}_4$$

find the matrix ${}_{\mathcal{C}}[L]_{\mathcal{B}}$ of L and use it to find $L(\vec{x})$ where $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$.

Solution

By definition, we have

$$[L(\vec{x})]_{\mathcal{C}} = {}_{\mathcal{C}}[L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 3 & 0 \\ 0 & 2 & 0 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ -6 \\ 0 \end{bmatrix}$$

This is the $\mathcal{C}$-coodrinate vector of $L(\vec{x})$, so by definition of coordinates,

$$L(\vec{x}) = 6\vec{w}_1 + 6\vec{w}_2 - 6\vec{w}_3$$

Matrix of a Linear mapping

Example

Let $T : \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ be the linear mapping defined by $T(a,b) = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, and let $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

Determine ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ and use it to calculate $T(\vec{v})$ where $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Solution

To find ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ we need to determine the $\mathcal{C}$-coordinates of the images of the vectors in $\mathcal{B}$ under T.

We have $T(2,-1) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$. We need to write this matrix as a linear combination of the vectors in $\mathcal{C}$.

That is, we need to find c_1, c_2, c_3, c_4 such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} = c_1 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

We row reduce the corresponding augmented matrix to get

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 3 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \Longrightarrow [T(2,-1)]_{\mathcal{C}} = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

Matrix of a Linear mapping

Example

Let $T : \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ be the linear mapping defined by $T(a,b) = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, and let $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

Determine ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ and use it to calculate $T(\vec{v})$ where $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Solution

Similarly, We find that

$$T(1,2) = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} = 4\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (-4)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

So, $[T(1,2)]_{\mathcal{C}} = \begin{bmatrix} 4 \\ 3 \\ -4 \\ 0 \end{bmatrix}$.

Hence,

$${}_{\mathcal{C}}[T]_{\mathcal{B}} = \begin{bmatrix} [T(2,-1)]_{\mathcal{C}} & [T(1,2)]_{\mathcal{C}} \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 1 & 3 \\ 2 & -4 \\ 0 & 0 \end{bmatrix}$$

Matrix of a Linear mapping

Example

Let $T : \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ be the linear mapping defined by $T(a,b) = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, and let $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

Determine ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ and use it to calculate $T(\vec{v})$ where $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Solution

Thus, we get

$$[T(\vec{v})]_{\mathcal{C}} = {}_{\mathcal{C}}[T]_{\mathcal{B}}[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} -2 & 4 \\ 1 & 3 \\ 2 & -4 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -16 \\ -7 \\ 16 \\ 0 \end{bmatrix}$$

Therefore, $T(\vec{v}) = (-16)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 16\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + (0)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 0 & 9 \end{bmatrix}$

Check:

We have $\vec{v} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix} - 3\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -8 \end{bmatrix}$. And $T(1,-8) = \begin{bmatrix} -7 & 0 \\ 0 & 9 \end{bmatrix}$ as before.

Matrix of a Linear mapping

Example

Let $T : \mathbb{R}^2 \to M_{2\times 2}(\mathbb{R})$ be the linear mapping defined by $T(a,b) = \begin{bmatrix} a+b & 0 \\ 0 & a-b \end{bmatrix}$.

Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$, and let $\mathcal{C} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$.

Determine ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ and use it to calculate $T(\vec{v})$ where $[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

Note:

- Although it is easier to solve for $T(\vec{v})$ using the latter method, the matrix ${}_{\mathcal{C}}[T]_{\mathcal{B}}$ helps us to better understand the linear mapping T and see its properties.
- For any linear mapping $L : \mathbb{V} \to \mathbb{W}$ we generally have $\text{Range}(L) \neq \text{Col}({}_{\mathcal{C}}[L]_{\mathcal{B}})$. In particular, the columnspace of a matrix is a subspace of $\mathbb{R}^n$, while $\text{Range}(L)$ can be a subspace of some other vector space (ie. $M_{2\times 2}(\mathbb{R})$).