Matrix of a Linear Mapping Continued

Last Lecture

We saw that if $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping and $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for $\mathbb{V}$ and $\mathcal{C}$ is a basis for $\mathbb{W}$, then the matrix of L with respect to bases $\mathcal{B}$ and $\mathcal{C}$ is defined by

$$_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \cdots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

It satisfies

$$[L(\vec{x})]_{\mathcal{C}} = {}_{\mathcal{C}}[L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}, \quad \text{for all } \vec{x} \in \mathbb{V}$$

In This Lecture

We will examine a special case where we have a linear operator L on a vector space $\mathbb{V}$ and we use the same basis $\mathcal{B}$ for the domain and codomain.

$\mathcal{B}$-Matrix of a Linear Mapping

Definition: Let $L : \mathbb{V} \to \mathbb{V}$ be a linear operator and let $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for $\mathbb{V}$. We define the matrix of L with respect to the basis $\mathcal{B}$, also called the $\mathcal{B}$-matrix of L, by

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & \cdots & [L(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

It satisfies

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}, \quad \text{for all } \vec{x} \in \mathbb{V}$$

$\mathcal{B}$-Matrix of a Linear Mapping

Example

Let $L : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear mapping defined by

$$L(a + bx + cx^2) = (a + b) + bx + (a + b + c)x^2$$

Find the matrix of L with respect to the basis $\mathcal{B} = \{1, x, x^2\}$.

Solution

We have

$$L(1) = 1 + x^2 \qquad L(x) = 1 + x + x^2 \qquad L(x^2) = x^2$$

Then, by definition,

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(1)]_{\mathcal{B}} & [L(x)]_{\mathcal{B}} & [L(x^2)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Check:

Take $a + bx + cx^2 \in P_2(\mathbb{R})$. Then $[a + bx + cx^2]_{\mathcal{B}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$.

$$[L(a + bx + cx^2)]_{\mathcal{B}} = [L]_{\mathcal{B}}[a + bx + cx^2]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b \\ a + b + c \end{bmatrix} = [L(a + bx + cx^2)]_{\mathcal{B}}$$

$\mathcal{B}$-Matrix of a Linear Mapping

Example

Let $\mathbb{U}$ be the subspace of $M_{2\times 2}(\mathbb{R})$ of upper triangular matrices and let $T : \mathbb{U} \to \mathbb{U}$ be the linear mapping defined by $T\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & b + c \\ 0 & a + b + c \end{bmatrix}$. Let $\mathcal{B}$ be the basis for $\mathbb{U}$ defined by $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$. Find the matrix of T with respect to the basis $\mathcal{B}$.

Solution

We have

$$T\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (2)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = (-1)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (0)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (2)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = (-2)\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + (-1)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + (4)\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

So, we get

$$[T]_{\mathcal{B}} = \begin{bmatrix} -1 & -1 & -2 \\ 0 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix}$$

$\mathcal{B}$-Matrix of a Linear Mapping

Example

Let $\mathbb{U}$ be the subspace of $M_{2\times 2}(\mathbb{R})$ of upper triangular matrices and let $T : \mathbb{U} \to \mathbb{U}$ be the linear mapping defined by $T\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a & b+c \\ 0 & a+b+c \end{bmatrix}$. Let $\mathcal{B}$ be the basis for $\mathbb{U}$ defined by $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$. Find the matrix of T with respect to the basis $\mathcal{B}$.

Solution

So, we get

$$[T]_{\mathcal{B}} = \begin{bmatrix} -1 & -1 & -2 \\ 0 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix}$$

Note the following:

- Finding the matrix of a linear mapping is a very algorithmic process. Master these types of questions so that you can do them quickly and correctly on tests.
- Many students have trouble working with coordinates. If you have such trouble, it is highly recommended that you take the time now to go back to your Linear Algebra 1 notes and review and practice this concept.

The check of this example is left as an important exercise.

Geometrically Natural Bases

Example

Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping defined by

$$L(x_1, x_2) = \left(\frac{9}{25} x_1 + \frac{12}{25} x_2, \frac{12}{25} x_1 + \frac{16}{25} x_2 \right)$$

Can you tell by just looking at this what the simple geometric interpretation of the mapping is?

Probably not. But we can use the $\mathcal{B}$-matrix of L to help us visualize the action of the mapping.

We have $[L] = \begin{bmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{bmatrix}$. How can we make this look nicer?

We can diagonalize!

Diagonalizing, we find that $P^{-1}[L]P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ where $P = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$.

Our work in Linear Algebra 1 shows us that $[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ where $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$.

We can now use this to get a clear geometric understanding of this mapping.

Geometrically Natural Bases

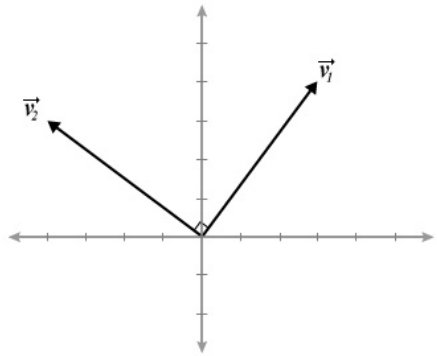
Example

$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$

By definition of $[L]_{\mathcal{B}}$ we have

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.



Geometrically Natural Bases

Example

$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$

By definition of $[L]_{\mathcal{B}}$ we have

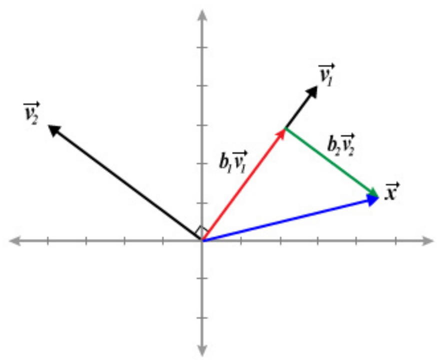
$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Let $\vec{x} = b_1\vec{v}_1 + b_2\vec{v}_2$ be any vector in $\mathbb{R}^2$. So, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.

Then,

$$[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$



Geometrically Natural Bases

Example

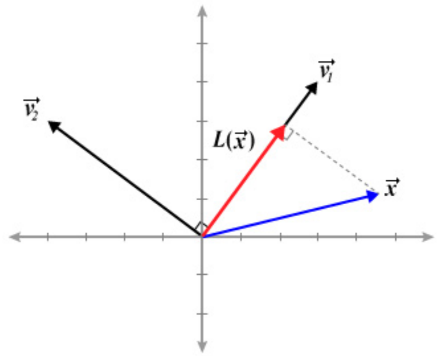
$[L]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \end{bmatrix} \right\}$

By definition of $[L]_{\mathcal{B}}$ we have

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

Let $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$.

Let $\vec{x} = b_1\vec{v}_1 + b_2\vec{v}_2$ be any vector in $\mathbb{R}^2$. So, $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$.



Then,

$$[L(\vec{x})]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \end{bmatrix}$$

Hence, by definition of coordinates, we have that $L(\vec{x}) = b_1\vec{v}_1$.

Thus, $L(\vec{x}) = L(b_1\vec{v}_1 + b_2\vec{v}_2) = b_1\vec{v}_1$. We recognize this mapping as the projection of $\vec{x}$ onto $\vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Geometrically Natural Bases

Definition: Let $L : \mathbb{V} \to \mathbb{V}$ be a linear operator. If $\mathcal{B}$ is a basis for $\mathbb{V}$ such that $[L]_{\mathcal{B}}$ is diagonal, then $\mathcal{B}$ is called a geometrically natural basis for L.

Note:

- The whole point of diagonalizing the standard matrix of a linear operator L is to find an associated geometrically natural basis $\mathcal{B}$.
- The vectors in $\mathcal{B}$ will always be the eigenvectors of the standard matrix $[L]$.
- We can use $[L]_{\mathcal{B}}$ to help us understand the mapping L.

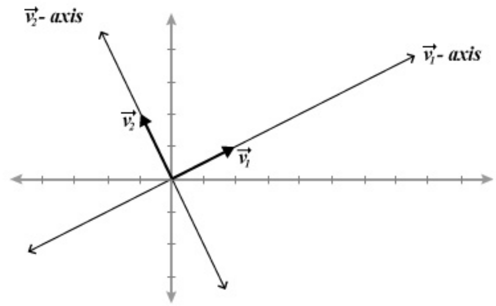
Geometrically Natural Bases

Example

Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ and define $L(\vec{x}) = A\vec{x}$. Describe geometrically the action of the linear mapping.

Solution

We find that the eigenvectors of A are $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with corresponding eigenvalue $\lambda_1 = 4$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ with corresponding eigenvalue $\lambda_2 = -1$.



Geometrically Natural Bases

Example

Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ and define $L(\vec{x}) = A\vec{x}$. Describe geometrically the action of the linear mapping.

Solution

We find that the eigenvectors of A are $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with corresponding eigenvalue $\lambda_1 = 4$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ with corresponding eigenvalue $\lambda_2 = -1$.

Hence, taking $P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ gives

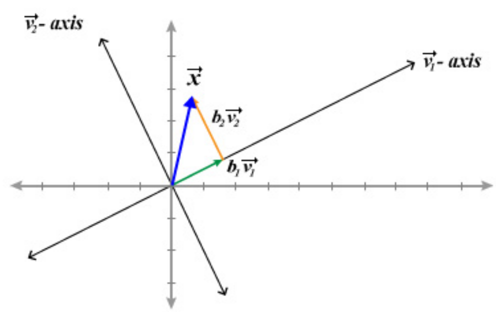
$$P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus, if we take $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, we get

$$[L]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

So, for any $\vec{x} = b_1\vec{v}_1 + b_2\vec{v}_2 \in \mathbb{R}^2$ we have

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 4b_1 \\ -b_2 \end{bmatrix}$$



Geometrically Natural Bases

Example

Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$ and define $L(\vec{x}) = A\vec{x}$. Describe geometrically the action of the linear mapping.

Solution

We find that the eigenvectors of A are $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ with corresponding eigenvalue $\lambda_1 = 4$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ with corresponding eigenvalue $\lambda_2 = -1$.

Hence, taking $P = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ gives

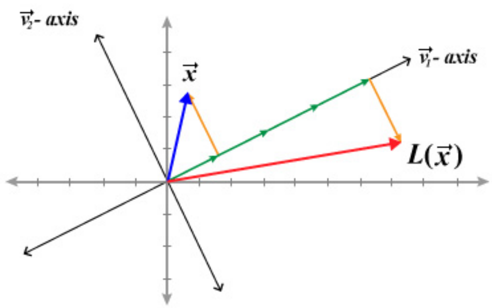
$$P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

Thus, if we take $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, we get

$$[L]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

So, for any $\vec{x} = b_1\vec{v}_1 + b_2\vec{v}_2 \in \mathbb{R}^2$ we have

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 4b_1 \\ -b_2 \end{bmatrix}$$



Therefore,

$$L(\vec{x}) = 4b_1\vec{v}_1 - b_2\vec{v}_2$$

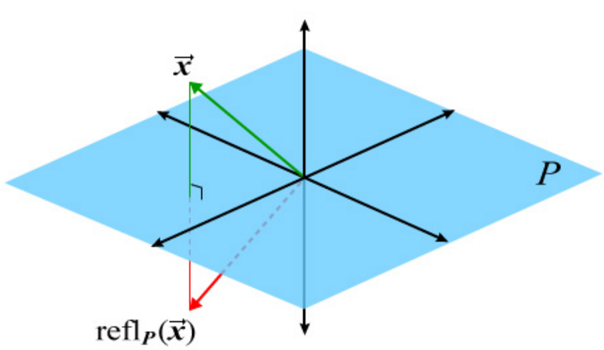
Hence, the linear mapping L takes a vector $\vec{x}$ and stretches it by a factor of 4 in the $\vec{v}_1$ direction and reflects it in the $\vec{v}_2$ direction.

Geometrically Natural Bases

Example

Let P be the plane in $\mathbb{R}^3$ with normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Find a geometrically natural basis $\mathcal{B}$ for the reflection $\text{refl}_P : \mathbb{R}^3 \to \mathbb{R}^3$ of a vector over the plane P, and find the $\mathcal{B}$-matrix of the reflection.

Solution



Geometrically Natural Bases

Example

Let P be the plane in $\mathbb{R}^3$ with normal vector $\vec{n} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$. Find a geometrically natural basis $\mathcal{B}$ for the reflection $\text{refl}_P : \mathbb{R}^3 \to \mathbb{R}^3$ of a vector over the plane P, and find the $\mathcal{B}$-matrix of the reflection.

Solution

Pick two linearly independent vectors on the plane $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ to form a basis for the plane.

Thus, our geometrically natural basis for $\mathbb{R}^3$ associated with refl_P is $\mathcal{B} = \{\vec{n}, \vec{v}_2, \vec{v}_3\}$.
We have

$$\begin{aligned} \text{refl}_P(\vec{n}) &= -\vec{n} = -1\vec{n} + 0\vec{v}_2 + 0\vec{v}_3 \\ \text{refl}_P(\vec{v}_2) &= \vec{v}_2 = 0\vec{n} + 1\vec{v}_2 + 0\vec{v}_3 \\ \text{refl}_P(\vec{v}_3) &= \vec{v}_3 = 0\vec{n} + 0\vec{v}_2 + 1\vec{v}_3 \end{aligned}$$

Hence,

$$[\text{refl}_P]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$