Maximizing and Minimizing Quadratic Forms

In some applications of quadratic forms we want to find the maximum and/or minimum of the quadratic form subject to a constraint.

Most of the time, it is possible to use a change of variables so that the constraint is $\|\vec{x}\| = 1$. That is, given a quadratic form $Q(\vec{x}) = \vec{x}^T A\vec{x}$ on $\mathbb{R}^n$ we want to find the maximum and minimum value of $Q(\vec{x})$ subject to $\|\vec{x}\| = 1$.

For ease, observe that we can rewrite the constraint as $1 = \|\vec{x}\|^2 = x_1^2 + \cdots + x_n^2$.

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Example

Find the maximum and minimum of $Q(x_1, x_2) = 2x_1^2 + 3x_2^2$ subject to the constraint $\|\vec{x}\| = 1$.

Solution

We have

$$Q(1,0) = 2 \qquad Q\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{5}{2}$$
$$Q\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{11}{4} \qquad Q(0,1) = 3$$

We have that

$$Q(x_1, x_2) = 2x_1^2 + 3x_2^2 \leq 3x_1^2 + 3x_2^2 = 3(x_1^2 + x_2^2) = 3$$

since $\|\vec{x}\|^2 = 1$ implies $x_1^2 + x_2^2 = 1$.

Hence, 3 is an upper bound for $Q(x_1, x_2)$, and $Q(0,1) = 3$, so 3 is the maximum value of $Q(x_1, x_2)$.

Similarly, we have that

$$Q(x_1, x_2) = 2x_1^2 + 3x_2^2 \geq 2x_1^2 + 2x_2^2 = 2(x_1^2 + x_2^2) = 2$$

So, 2 is a lower bound for $Q(x_1, x_2)$, and $Q(1,0) = 2$, so the minimum value is 2.

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Example

Find the maximum and minimum of $Q(x_1, x_2, x_3) = 4x_1^2 + 4x_1x_2 + 4x_1x_3 + 4x_2^2 + 4x_2x_3 + 4x_3^2$ subject to $x_1^2 + x_2^2 + x_3^2 = 1$.

Solution

The symmetric matrix corresponding to Q is $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. We can find that the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2$, and $\lambda_3 = 8$. Thus, by Theorem 10.3.1 there exists an orthogonal matrix $P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}$ such that the change of variables $\vec{y} = P^T\vec{x}$ brings Q into diagonal form

$$Q(\vec{x}) = 2y_1^2 + 2y_2^2 + 8y_3^2$$

We have

$$Q(\vec{x}) = 2y_1^2 + 2y_2^2 + 8y_3^2 \leq 8y_1^2 + 8y_2^2 + 8y_3^2 = 8(y_1^2 + y_2^2 + y_3^2)$$

Using the change of variables and that P is orthogonal, we have that

$$\|\vec{y}\| = \|P^T\vec{x}\| = \|\vec{x}\| = 1$$

Thus, we do have $y_1^2 + y_2^2 + y_3^2 = 1$, and so $Q(\vec{x}) \leq 8$ for $\|\vec{x}\| = 1$.
Thus, 8 is upper bound for $Q(\vec{x})$.
Taking $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ gives $Q(\vec{y}) = 8$, so 8 is indeed the maximum.

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Example

Find the maximum and minimum of $Q(x_1, x_2, x_3) = 4x_1^2 + 4x_1x_2 + 4x_1x_3 + 4x_2^2 + 4x_2x_3 + 4x_3^2$ subject to $x_1^2 + x_2^2 + x_3^2 = 1$.

Solution

Similarly

$$Q(\vec{x}) \geq 2y_1^2 + 2y_2^2 + 2y_3^2 = 2(y_1^2 + y_2^2 + y_3^2) = 2$$

Thus, a lower bound of $Q(\vec{x})$ is 2, and this value is achieved when $\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. So, 2 is the minimum.

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Example

Find the maximum and minimum of $Q(x_1, x_2, x_3) = 4x_1^2 + 4x_1x_2 + 4x_1x_3 + 4x_2^2 + 4x_2x_3 + 4x_3^2$ subject to $x_1^2 + x_2^2 + x_3^2 = 1$.

Solution

We get the diagonal form of $Q(\vec{x})$ by applying the change of variables $\vec{y} = P^T\vec{x}$, which can be rearranged to $\vec{x} = P\vec{y}$.

Since $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ gives the maximum value of $Q(\vec{x})$, we have that

$$\vec{x} = P\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \vec{v}_3$$

As P is an orthogonal matrix that diagonalizes A, $\vec{v}_3$ is a unit eigenvector of A corresponding to the eigenvalue 8.

Repeating this procedure, we see that the minimum 2 is obtained when

$$\vec{x} = P\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v}_1$$

a unit eigenvector of A corresponding to the eigenvalue 2.

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Theorem 10.5.1

If $Q(\vec{x}) = \vec{x}^T A\vec{x}$ is a quadratic form with symmetric matrix A, then the maximum of $Q(\vec{x})$ subject to $\|\vec{x}\| = 1$ is the largest eigenvalue of A and the minimum of $Q(\vec{x})$ subject to $\|\vec{x}\| = 1$ is the smallest eigenvalue of A. Moreover, the maximum and minimum occur at unit eigenvectors of A corresponding to the respective eigenvalues.

Proof

Since A is symmetric, by the Principal Axis Theorem we can find an orthogonal matrix $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ such that $P^T AP = \text{diag}(\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$.
Using Theorem 10.3.1, we get that the change of variables $\vec{y} = P^T\vec{x}$ brings $Q(\vec{x})$ into diagonal form

$$Q(\vec{x}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

Then, $\|\vec{y}\| = \|P^T\vec{x}\| = \|\vec{x}\| = 1$ since P is orthogonal.

Maximizing and Minimizing Quadratic Forms

Theorem 10.5.1

If $Q(\vec{x}) = \vec{x}^T A\vec{x}$ is a quadratic form with symmetric matrix A, then the maximum of $Q(\vec{x})$ subject to $||\vec{x}|| = 1$ is the largest eigenvalue of A and the minimum of $Q(\vec{x})$ subject to $||\vec{x}|| = 1$ is the smallest eigenvalue of A. Moreover, the maximum and minimum occur at unit eigenvectors of A corresponding to the respective eigenvalues.

Proof

Since $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ we now see that

$$Q(\vec{x}) \leq \lambda_1 y_1^2 + \cdots + \lambda_1 y_n^2 = \lambda_1(y_1^2 + \cdots + y_n^2) = \lambda_1$$

and

$$Q(\vec{x}) \geq \lambda_n y_1^2 + \cdots + \lambda_n y_n^2 = \lambda_n(y_1^2 + \cdots + y_n^2) = \lambda_n$$

Moreover, we have $Q(\vec{x}) = \lambda_1$ at $\vec{y} = \vec{e}_1$ which gives $\vec{x} = P\vec{e}_1 = \vec{v}_1$ a unit eigenvector of A corresponding to λ_1.
Similarly, $Q(\vec{x}) = \lambda_n$ at $\vec{y} = \vec{e}_n$ which gives $\vec{x} = P\vec{e}_n = \vec{v}_n$ a unit eigenvector of A corresponding to λ_n.
Therefore, the maximum of $Q(\vec{x})$ is λ_1 and occurs at $\vec{x} = \vec{v}_1$, and the minimum of $Q(\vec{x})$ is λ_n and occurs at $\vec{x} = \vec{v}_n$. $\square$

Note: The vector where the maximum and minimum of $Q(\vec{x})$ occurs is not unique.