Quadratic Forms

So far in linear algebra, we have been looking at linear forms $a_1x_1 + \cdots + a_nx_n$.
However, a very important topic in many areas of mathematics is that of a quadratic form. That is, an equation which is a linear combination of all possible terms x_ix_j for $1 \leq i \leq j \leq n$.

For example, for two variables x_1, x_2 we have

$$Q(x_1, x_2) = a_1x_1^2 + a_2x_1x_2 + a_3x_2^2, \quad a_1, a_2, a_3 \in \mathbb{R}$$

For three variables x_1, x_2, x_3 we get

$$Q(x_1, x_2, x_3) = a_1x_1^2 + a_2x_1x_2 + a_3x_1x_3 + a_4x_2^2 + a_5x_2x_3 + a_6x_3^2, \quad a_1, \dots, a_6 \in \mathbb{R}$$

Although quadratic forms are not linear, they do have a connection to linear algebra through matrix multiplication.

$$\begin{aligned} \vec{x}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x} &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} (ax_1 + cx_2) & (bx_1 + dx_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= (ax_1 + cx_2)x_1 + (bx_1 + dx_2)x_2 \\ &= ax_1^2 + (b + c)x_1x_2 + dx_2^2 \end{aligned}$$

We can use this relationship to make a more useful definition for a quadratic form on $\mathbb{R}^n$.

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Definition: We define a quadratic form on $\mathbb{R}^n$ with corresponding matrix A by

$$Q(\vec{x}) = \vec{x}^T A\vec{x}, \quad \vec{x} \in \mathbb{R}^n$$

Notice that there is a connection between the dot product and quadratic forms:

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \vec{x} \cdot (A\vec{x}) = (A\vec{x}) \cdot \vec{x} = (A\vec{x})^T\vec{x} = \vec{x}^T A^T \vec{x}$$

Hence, A and A^T give the same quadratic form.

But note that this does *not* imply that $A = A^T$.

However, it can be shown that every quadratic form can be written as $Q(\vec{x}) = \vec{x}^T A\vec{x}$ where A is symmetric.
Moreover, each symmetric matrix A *uniquely* determines a quadratic form.

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Example

Find the quadratic form corresponding to $A = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$.

Solution

By definition, we have

$$\begin{aligned} Q(x_1, x_2) &= \vec{x}^T A \vec{x} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2x_1 + 3x_2 \\ 3x_1 - x_2 \end{bmatrix} \\ &= 2x_1^2 + 6x_1x_2 - x_2^2 \end{aligned}$$

Example

Find the symmetric matrix which corresponds to the quadratic form

$$Q(x_1, x_2) = 4x_1^2 - 2x_1x_2 + 7x_2^2$$

Solution

	x_1	x_2
x_1	4	-1
x_2	-1	7

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Example

Find the symmetric matrix which corresponds to the quadratic form

$$Q(x_1, x_2) = 4x_1^2 - 2x_1x_2 + 7x_2^2$$

Solution

$$A = \begin{bmatrix} 4 & -1 \\ -1 & 7 \end{bmatrix}$$

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Quadratic Forms

Example

Find the symmetric matrix corresponding to the quadratic form

$$Q(x_1, x_2, x_3) = 2x_1^2 + 4x_1x_2 + 2x_1x_3 - 3x_2^2 - 6x_2x_3 + 5x_3^2$$

Solution

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & -3 & -3 \\ 1 & -3 & 5 \end{bmatrix}$$

Note: It is very important to be able to quickly convert back and forth between a quadratic form and its corresponding symmetric matrix.

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Definition: If the symmetric matrix corresponding to the quadratic form $Q(\vec{x}) = \vec{x}^T A\vec{x}$ is diagonal, then we say that $Q(\vec{x})$ is in diagonal form.

We now prove that orthogonally diagonalizing a symmetric matrix A corresponds to performing a change of variables on the quadratic form $Q(\vec{x}) = \vec{x}^T A\vec{x}$ that brings $Q(\vec{x})$ into diagonal form.

Theorem 10.3.1

If $Q(\vec{x}) = \vec{x}^T A\vec{x}$ is a quadratic form in n variables with corresponding symmeric matrix A and P is an orthogonal matrix such that $P^T AP = D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, then performing the change of variables $\vec{y} = P^T\vec{x}$ gives

$$Q(\vec{x}) = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

Proof

Since P is orthogonal, we have that $\vec{y} = P^T\vec{x} \Rightarrow \vec{x} = P\vec{y}$. Therefore, we have

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = (P\vec{y})^T A(P\vec{y}) = \vec{y}^T (P^T AP)\vec{y} = \vec{y}^T D\vec{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

□

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Example

Find new variables y_1, y_2, y_3, y_4 such that

$$Q(x_1,x_2,x_3,x_4) = 3x_1^2 + 2x_1x_2 - 10x_1x_3 + 10x_1x_4 + 3x_2^2 + 10x_2x_3 - 10x_2x_4 + 3x_3^2 + 2x_3x_4 + 3x_4^2$$

has diagonal form.

Solution

Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ and $A = \begin{bmatrix} 3 & 1 & -5 & 5 \\ 1 & 3 & 5 & -5 \\ -5 & 5 & 3 & 1 \\ 5 & -5 & 1 & 3 \end{bmatrix}$.

Doing the calculations, we find that $P = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}$ gives $P^TAP = \begin{bmatrix} 12 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$

According to the theorem, we take $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = P^T\vec{x} = \begin{bmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ \frac{1}{2}x_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_3 - \frac{1}{2}x_4 \end{bmatrix}$.

And this gives

$$Q(x_1,x_2,x_3,x_4) = 12y_1^2 - 8y_2^2 + 4y_3^2 + 4y_4^2$$

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In summary, for the given quadratic form $Q(x_1,x_2,x_3,x_4)$, we found that

$$P = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & 1/2 & -1/2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = P^T\vec{x} \Rightarrow Q(x_1,x_2,x_3,x_4) = 12y_1^2 - 8y_2^2 + 4y_3^2 + 4y_4^2$$

Notes:

(1) The orthonormal eigenvectors we used to construct P are called the principal axes of A.

(2) By changing the order of the eigenvectors in P, we also change the order of the eigenvalues in $D = P^TAP$ and hence the coefficients of the corresponding y_i.

For example,

$$P = \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}, \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = P^T\vec{x} \Rightarrow Q(x_1,x_2,x_3,x_4) = -8y_1^2 + 4y_2^2 + 4y_3^2 + 12y_4^2$$

Notice, that since we can pick any vector $\vec{y} \in \mathbb{R}^4$, this choice of P gives us exactly the same set of values for the quadratic form as with our original choice of P.

You can also think of just doing another change of variables $z_1 = y_2$, $z_2 = y_3$, $z_3 = y_4$ and $z_4 = y_1$.

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