Singular Values

We have now seen that symmetric matrices can be orthogonally diagonalized.
If we don't have a symmetric matrix, we still may be able to diagonalize it.
We have also seen that if a matrix has all real eigenvalues, then we can triangularize it.

To do all that was mentioned we must have a square matrix, so what do we do if we get an $m \times n$ matrix with $m \neq n$?
We will figure out the answer to this question in the next couple of lectures.

Singular Values

Example

Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & -2 \end{bmatrix}$ and let L be the linear mapping $L(\vec{x}) = A\vec{x}$. What is the maximum and minimum of $\|L(\vec{x})\|$ subject to the constraint $\|\vec{x}\| = 1$?

Solution

We get

$$\|L(\vec{x})\|^2 = \|A\vec{x}\|^2 = (A\vec{x}) \cdot (A\vec{x}) = (A\vec{x})^T(A\vec{x}) = \vec{x}^T(A^TA)\vec{x}$$

We can solve this easily by using Theorem 10.5.1.
Thus, the maximum occurs at the largest eigenvalue of A^TA and the minimum at the smallest eigenvalue of A^TA.
We have

$$A^TA = \begin{bmatrix} 6 & -2 \\ -2 & 9 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 10$, $\lambda_2 = 5$.
Thus, the largest eigenvalue of A^TA is 10, so the maximum of $\|L(\vec{x})\|^2$ is 10.
To find the maximum of $\|L(\vec{x})\|$ we need to take the square root of $\|L(\vec{x})\|^2$.

Singular Values

Example

Let $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \\ 1 & -2 \end{bmatrix}$ and let L be the linear mapping $L(\vec{x}) = A\vec{x}$. What is the maximum and minimum of $\|L(\vec{x})\|$ subject to the constraint $\|\vec{x}\| = 1$?

Solution

So, we get the maximum of $\|L(\vec{x})\|$ subject to $\|\vec{x}\| = 1$ is $\sqrt{10}$.

Similarly, the minimum of $\|L(\vec{x})\|$ subject to $\|\vec{x}\| = 1$ is $\sqrt{5}$.

We find that a unit eigenvector for $\lambda_1 = 10$ is $\vec{v}_1 = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, and $\vec{v}_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ is a unit eigenvector for $\lambda_2 = 5$.

We easily verify that

$$\|L(\vec{v}_1)\| = \left\| \begin{bmatrix} 3/\sqrt{5} \\ 4/\sqrt{5} \\ -\sqrt{5} \end{bmatrix} \right\| = \sqrt{10} \quad \|L(\vec{v}_2)\| = \left\| \begin{bmatrix} 4/\sqrt{5} \\ -3/\sqrt{5} \\ 0 \end{bmatrix} \right\| = \sqrt{5}$$

Singular Values

Theorem 10.6.1

If A is an $m \times n$ matrix and $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A^TA with corresponding unit eigenvectors $\vec{v}_1, \dots, \vec{v}_n$, then $\lambda_1, \dots, \lambda_n$ are all non-negative and

$$\|A\vec{v}_i\| = \sqrt{\lambda_i}$$

Proof

For $1 \le i \le n$ we are assuming that $\vec{v}_i$ is an eigenvector of A^TA, so this means that $A^TA\vec{v}_i = \lambda_i\vec{v}_i$. Thus, we get

$$\|A\vec{v}_i\|^2 = (A\vec{v}_i) \cdot (A\vec{v}_i) = (A\vec{v}_i)^T A\vec{v}_i = v_i^T A^T A\vec{v}_i = \vec{v}_i^T(\lambda_i\vec{v}_i) = \lambda_i(\vec{v}_i \cdot \vec{v}_i) = \lambda_i$$

Therefore, λ_i is equal to the non-negative number $\|A\vec{v}_i\|^2$, and the result follows.

□

Note: From the example before the Theorem, the square root of the eigenvalues of A^TA are the maximum and minimum of values of $\|A\vec{x}\|$ subject to $\|\vec{x}\| = 1$. So, these are behaving like the eigenvalues of a symmetric matrix.

Singular Values

Definition: The singular values $\sigma_1, \dots, \sigma_n$ of an $m \times n$ matrix A are the square roots of the eigenvalues of A^TA arranged so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$.

Note: By definition, we **must** arrange the singular values from greatest to least.
The reasoning for this is similar to what we saw in Theorem 10.5.1: we want to know where the largest and smallest ones are. In particular, we need to ensure all of the 0 singular values are at the end.

Example

Find the singular values of $A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$.

Solution

We have $A^TA = \begin{bmatrix} 2 & -1 & -2 \\ -1 & 5 & 1 \\ -2 & 1 & 2 \end{bmatrix}$.

We find that the eigenvalues of A^TA are 0, 6, and 3.
Thus, the singular values of A are $\sigma_1 = \sqrt{6}$, $\sigma_2 = \sqrt{3}$, and $\sigma_3 = 0$.

Singular Values

We know that the number of non-zero eigenvalues of a square matrix A equals the rank of A.
In the example and the check-in, we got that the number of non-zero singular values of A equaled the rank of A.
To prove that this is always the case, we just need to prove that the rank of A^TA equals the rank of A.

Lemma 10.6.2

If A is an $m \times n$ matrix, then $\text{Null}(A^TA) = \text{Null}(A)$.

Proof

Assume $\vec{x} \in \text{Null}(A)$, then $A\vec{x} = \vec{0}$ and $A^TA\vec{x} = A^T\vec{0} = \vec{0}$, hence $\vec{x} \in \text{Null}(A^TA)$.
So, $\text{Null}(A) \subseteq \text{Null}(A^TA)$.

Assume $\vec{x} \in \text{Null}(A^TA)$.
If $A^TA\vec{x} = \vec{0}$, then

$$\|A\vec{x}\|^2 = (A\vec{x}) \cdot (A\vec{x}) = (A\vec{x})^T(A\vec{x}) = \vec{x}^TA^TA\vec{x} = \vec{x}^T\vec{0} = 0$$

Hence, $A\vec{x} = \vec{0}$ and so $\vec{x} \in \text{Null}(A)$.
Thus, $\text{Null}(A^TA) \subseteq \text{Null}(A)$, and consequently $\text{Null}(A) = \text{Null}(A^TA)$.

$\square$

Singular Values

Theorem 10.6.3

If A is an $m \times n$ matrix, then $\text{rank}(A^T A) = \text{rank}(A)$.

Proof

Using Lemma 10.6.2 and the Dimension Theorem, we get

$$\text{rank}(A^T A) = n - \dim(\text{Null}(A^T A)) = n - \dim(\text{Null}(A)) = \text{rank}(A)$$

□

Corollary 10.6.4

If A is an $m \times n$ matrix and $\text{rank}(A) = r$, then A has r non-zero singular values.

We see that the singular values of a matrix A have a lot of the same properties that eigenvalues have.
This should not be surprising, since we have defined singular values in terms of eigenvalues: they are the square roots of the eigenvalues of $A^T A$.