The Cayley-Hamilton Theorem

Consider $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. The characteristic polynomial is

$$C(\lambda) = \lambda^2 - 3\lambda - 4$$

For our matrix A we get

$$A^2 - 3A - 4I = \begin{bmatrix} 7 & 6 \\ 9 & 10 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So, A is a root of its characteristic polynomial!

We now prove that this is true for every matrix A.

The Cayley-Hamilton Theorem

Theorem 11.6.1 - The Cayley Hamilton Theorem

If $C(\lambda)$ is the characteristic polynomial of an $n \times n$ matrix A, then $C(A) = O_{n,n}$.

Proof

Let $C(\lambda) = (-1)^n \lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$.

We now consider the polynomial

$$C(X) = (-1)^n X^n + c_{n-1}X^{n-1} + \cdots + c_1 X + c_0 I$$

whose argument is a matrix X.

By Schur's theorem, there exists a unitary matrix U and upper triangular matrix T such that $U^* A U = T$.

Since the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A are the diagonal entries of T we have $T = \begin{bmatrix} \lambda_1 & t_{12} & \cdots & t_{1n} \\ 0 & \lambda_2 & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$.

Moreover, we know that the roots of the characteristic polynomial are the eigenvalues of A. Thus, we can factor $C(X)$ as

$$C(X) = (X - \lambda_1 I)(X - \lambda_2 I) \cdots (X - \lambda_n I)$$

Therefore,

$$C(T) = (T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)$$

We will show that $C(T)$ is the zero matrix by showing that the first k-columns of $(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_k I)$ only contain zeros for $1 \leq k \leq n$.

The Cayley-Hamilton Theorem

Theorem 11.6.1 - The Cayley Hamilton Theorem

If $C(\lambda)$ is the characteristic polynomial of an $n \times n$ matrix A, then $C(A) = O_{n,n}$.

Proof

Observe that the first column of $T - \lambda_1 I$ is zero since the first column of T is $\begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

Assume that all entries of the first k-columns of $(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_k I)$ only contain zeros.

Then, it is easy to verify that the first $k+1$ columns of $(T - \lambda_1 I) \cdots (T - \lambda_k I)(T - \lambda_{k+1} I)$ only contain zeros.

Thus, by induction, $C(T) = O_{n,n}$.

We now observe that since $A = UTU^*$ we have

$$\begin{aligned} C(A) &= C(UTU^*) \\ &= (-1)^n (UTU^*)^n + \cdots + c_1(UTU^*) + c_0 I \\ &= (-1)^n UT^n U^* + \cdots + c_1 UTU^* + c_0 UU^* \\ &= U\big[(-1)^n T^n + \cdots + c_1 T + c_0 I\big] U^* \\ &= UC(T)U^* \\ &= O_{n,n} \end{aligned}$$

as required.

□

The Cayley-Hamilton Theorem

A natural question to ask is if the converse of the Cayley-Hamilton Theorem is true.

That is, if $p(x)$ is a polynomial such that $p(A) = O_{n,n}$, then must $p(x)$ be the characteristic polynomial of A?

Unfortunately, the answer is no. The converse of the Cayley-Hamilton Theorem is not true.

For example, the matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ satisfies $A^2 = O_{3,3}$, but the characteristic polynomial of A is $C(\lambda) = \lambda^3$.

The Cayley-Hamilton Theorem

What does the Cayley-Hamilton Theorem say about the inverse of a matrix?
Assume that A is an invertible matrix with characteristic polynomial

$$C(\lambda) = (-1)^n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0$$

Observe that since A is invertible, we must have $c_0 \neq 0$.
Then, the Cayley-Hamilton theorem gives

$$(-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_1 A + c_0 I = O_{n,n}$$

Hence we can solve the equation for I. We get

$$I = -\frac{1}{c_0}\left((-1)^n A^n + c_{n-1}A^{n-1} + \cdots + c_2A^2 + c_1A\right) = -\frac{1}{c_0}\left((-1)^n A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_2A + c_1I\right)A$$

Thus,

$$A^{-1} = -\frac{1}{c_0}\left((-1)^n A^{n-1} + c_{n-1}A^{n-2} + \cdots + c_2A + c_1I\right)$$

and so the Cayley-Hamilton Theorem gives us a formula for the inverse of an invertible matrix A as a linear combination of powers of A.

The Cayley-Hamilton Theorem

Example

Let $A = \begin{bmatrix} 1 & 1 & -3 \\ 2 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$. Write the inverse of A as a linear combination of powers of A.

Solution

We have

$$C(\lambda) = \begin{vmatrix} 1-\lambda & 1 & -3 \\ 2 & -\lambda & 2 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \lambda^3 - 2\lambda^2 + 8$$

Thus, by the Cayley-Hamilton Theorem, we have

$$\begin{aligned} A^3 - 2A^2 + 8I &= O_{3,3} \\ A^3 - 2A^2 &= -8I \\ -\tfrac{1}{8}A^3 + \tfrac{1}{4}A^2 &= I \\ A\left(-\tfrac{1}{8}A^2 + \tfrac{1}{4}A\right) &= I \\ A^{-1} &= -\tfrac{1}{8}A^2 + \tfrac{1}{4}A \end{aligned}$$