The Fundamental Theorem of Linear Algebra

We have now seen that if $\mathbb{W}$ is a subspace of an inner product space $\mathbb{V}$, then every vector $\vec{v} \in \mathbb{V}$ can be written as

$$\vec{v} = \vec{w} + \vec{x}$$

where $\vec{w} \in \mathbb{W}$ and $\vec{x} \in \mathbb{W}^{\perp}$.

In This Lecture

- We will invent some notation for this idea.
- We will use our work with orthogonal complements and fundamental subspaces to prove the Fundamental Theorem of Linear Algebra.

Direct Sums

Definition: Let $\mathbb{U}$ and $\mathbb{W}$ be subspaces of a vector space $\mathbb{V}$ such that $\mathbb{U} \cap \mathbb{W} = \{\vec{0}\}$. We define the direct sum of $\mathbb{U}$ and $\mathbb{W}$ by

$$\mathbb{U} \oplus \mathbb{W} = \{\vec{u} + \vec{w} \mid \vec{u} \in \mathbb{U}, \vec{w} \in \mathbb{W}\}$$

Note:

- Before taking the direct sum of two subspaces, you must check that their intersection only contains $\vec{0}$
- Whenever we write $\mathbb{U} \oplus \mathbb{W}$, we are implying that $\mathbb{U} \cap \mathbb{W} = \{\vec{0}\}$

Direct Sums

Example

Let $\mathbb{U} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$ and $\mathbb{W} = \operatorname{Span}\left\{\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}$ be subspaces of $\mathbb{R}^3$. What is $\mathbb{U} \oplus \mathbb{W}$?

Solution

By definition, we have that

$$\begin{aligned}\mathbb{U} \oplus \mathbb{W} &= \{\vec{u} + \vec{w} \mid \vec{u} \in \mathbb{U}, \vec{w} \in \mathbb{W}\} \\ &= \left\{ s\begin{bmatrix}1\\0\\0\end{bmatrix} + t\begin{bmatrix}0\\1\\2\end{bmatrix} \mid s, t \in \mathbb{R}\right\} \\ &= \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix}\right\}\end{aligned}$$

Direct Sums

Theorem 9.5.1

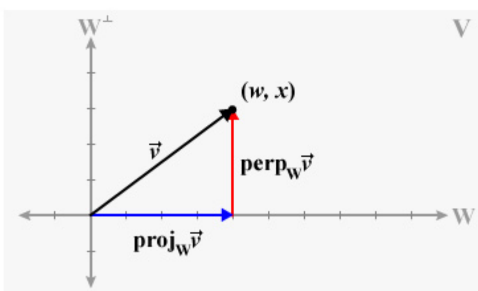
If $\mathbb{U}$ and $\mathbb{W}$ are subspaces of a vector space $\mathbb{V}$, then $\mathbb{U} \oplus \mathbb{W}$ is a subspace of $\mathbb{V}$. Moreover, if $\{\vec{u}_1, \dots, \vec{u}_k\}$ is a basis for $\mathbb{U}$ and $\{\vec{w}_1, \dots, \vec{w}_\ell\}$ is a basis for $\mathbb{W}$, then $\{\vec{u}_1, \dots, \vec{u}_k, \vec{w}_1, \dots, \vec{w}_\ell\}$ is a basis for $\mathbb{U} \oplus \mathbb{W}$.

The proof is left as an exercise.

Theorem 9.5.2

If $\mathbb{V}$ is a finite dimensional inner product space and $\mathbb{W}$ is a subspace of $\mathbb{V}$, then

$$\mathbb{W} \oplus \mathbb{W}^\perp = \mathbb{V}$$



Direct Sums

Example

Find a basis for each of the four fundamental subspaces of $A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 6 & 1 & -1 \\ -2 & -4 & 2 & 6 \end{bmatrix}$.

Solution

Row reduce A to its RREF $R = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Then, a basis for $\text{Row}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

A basis for $\text{Col}(A)$ is $\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

And a basis for $\text{Null}(A)$ is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Row reduce A^T to its RREF $\begin{bmatrix} 1 & 0 & -8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Thus, a basis for $\text{Null}(A^T)$ is $\left\{ \begin{bmatrix} 8 \\ -2 \\ 1 \end{bmatrix} \right\}$.

Notes:
$\text{Row}(A) \oplus \text{Null}(A) = \mathbb{R}^4$
$\text{Col}(A) \oplus \text{Null}(A^T) = \mathbb{R}^3$
$\text{Row}(A)^\perp = \text{Null}(A)$
$\text{Col}(A)^\perp = \text{Null}(A^T)$

The Fundamental Theorem of Linear Algebra

Theorem 9.5.3 - The Fundamental Theorem of Linear Algebra

If A is an $m \times n$ matrix, then $\text{Col}(A)^\perp = \text{Null}(A^T)$, $\text{Row}(A)^\perp = \text{Null}(A)$. In particular,

$$\mathbb{R}^n = \text{Row}(A) \oplus \text{Null}(A) \quad \text{and} \quad \mathbb{R}^m = \text{Col}(A) \oplus \text{Null}(A^T)$$

Proof

We will start by proving that $\text{Row}(A)^\perp = \text{Null}(A)$.

Let $A = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix}$.

Let $\vec{x} \in \text{Row}(A)^\perp$. Then $\vec{v}_i \cdot \vec{x} = 0$ for $1 \leq i \leq m$.
Thus, we have

$$A\vec{x} = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vdots \\ \vec{v}_m \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hence, $\vec{x} \in \text{Null}(A)$.
Therefore, $\text{Row}(A)^\perp \subseteq \text{Null}(A)$

The Fundamental Theorem of Linear Algebra

Theorem 9.5.3 - The Fundamental Theorem of Linear Algebra

If A is an $m \times n$ matrix, then $\mathrm{Col}(A)^{\perp} = \mathrm{Null}(A^T)$, $\mathrm{Row}(A)^{\perp} = \mathrm{Null}(A)$. In particular,

$$\mathbb{R}^n = \mathrm{Row}(A) \oplus \mathrm{Null}(A) \quad \text{and} \quad \mathbb{R}^m = \mathrm{Col}(A) \oplus \mathrm{Null}(A^T)$$

Proof

We will start by proving that $\mathrm{Row}(A)^{\perp} = \mathrm{Null}(A)$.

Let $A = \begin{bmatrix} \vec{v}_1^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix}$.

On the other hand, let $\vec{y} \in \mathrm{Null}(A)$.
Then we have

$$\vec{0} = A\vec{y} = \begin{bmatrix} \vec{v}_1 \cdot \vec{y} \\ \vdots \\ \vec{v}_m \cdot \vec{y} \end{bmatrix}$$

Consequently, by Theorem 9.4.1, $\vec{y} \in \mathrm{Row}(A)^{\perp}$.
Therefore, $\mathrm{Null}(A) \subseteq (\mathrm{Row}(A))^{\perp}$ and so $\mathrm{Null}(A) = (\mathrm{Row}(A))^{\perp}$.

The Fundamental Theorem of Linear Algebra

Theorem 9.5.3 - The Fundamental Theorem of Linear Algebra

If A is an $m \times n$ matrix, then $\mathrm{Col}(A)^{\perp} = \mathrm{Null}(A^T)$, $\mathrm{Row}(A)^{\perp} = \mathrm{Null}(A)$. In particular,

$$\mathbb{R}^n = \mathrm{Row}(A) \oplus \mathrm{Null}(A) \quad \text{and} \quad \mathbb{R}^m = \mathrm{Col}(A) \oplus \mathrm{Null}(A^T)$$

Proof

Applying what we just proved to A^T, we get

$$(\mathrm{Col}\, A)^{\perp} = (\mathrm{Row}(A^T))^{\perp} = \mathrm{Null}(A^T)$$

The remaining part of the theorem follows from Theorem 9.5.2.

$\square$

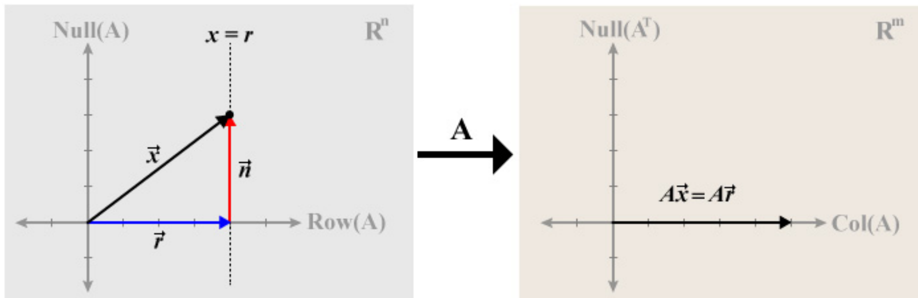
The Fundamental Theorem of Linear Algebra

Notes on the Fundamental Theorem

Since $(\text{Row}(A))^\perp = \text{Null}(A)$, we have from our work with orthogonal complements that

$$\dim(\text{Null}(A)) = \dim((\text{Row}(A))^\perp) = n - \dim \text{Row}(A) = n - \text{rank}(A)$$

Hence, the Fundamental Theorem of Linear Algebra immediately implies the Dimension Theorem.



Any $\vec{x} \in \mathbb{R}^n$ can be written as $\vec{x} = \vec{r} + \vec{n}$ where $\vec{r} \in \text{Row}(A)$ and $\vec{n} \in \text{Null}(A)$.Multiplying by A, we get

$$A\vec{x} = A(\vec{r} + \vec{n}) = A\vec{r} + A\vec{n} = A\vec{r} + \vec{0} = A\vec{r}$$

This implies that a consistent system $A\vec{x} = \vec{b}$ has a unique solution only if $\text{Null}(A) = \{\vec{0}\}$.