The Gram-Schmidt Procedure

Last Lecture

- We saw that orthonormal bases are very useful.

In This Lecture

- We will learn how to use the Gram-Schmidt procedure to turn a basis $\{\vec{w}_1, \ldots, \vec{w}_k\}$ for a subspace $\mathbb{W}$ of an inner product space $\mathbb{V}$ into an orthogonal basis $\{\vec{v}_1, \ldots, \vec{v}_k\}$ for $\mathbb{W}$.

The Gram-Schmidt Procedure

First consider the case where $\mathbb{W}$ is a 1-dimensional subspace of an inner product space $\mathbb{V}$. Then, we have a basis $\{\vec{w}_1\}$ for $\mathbb{W}$. Since $\{\vec{w}_1\}$ is an orthogonal basis for $\mathbb{W}$, we can take $\vec{v}_1 = \vec{w}_1$.

Next, consider the case where $\mathbb{W}$ is a 2-dimensional subspace of $\mathbb{V}$ with basis $\{\vec{w}_1, \vec{w}_2\}$. Starting as in the case above we take $\vec{v}_1 = \vec{w}_1$. We now need to prove that there must exist a vector $\vec{v}_2$ in $\mathbb{W}$ that is orthogonal to $\vec{v}_1$.

Assume there is a vector $\vec{v}_2$ in $\mathbb{W}$ such that $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for $\mathbb{W}$. Since $\vec{w}_2$ is in $\mathbb{W}$, we can write it as a linear combination of the orthogonal basis vectors. Using our work from last lecture we get

$$\vec{w}_2 = \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\langle \vec{w}_2, \vec{v}_2 \rangle}{\|\vec{v}_2\|^2} \vec{v}_2$$

Since $\vec{w}_2$ is not a scalar multiple of $\vec{w}_1 = \vec{v}_1$ we must have have $\langle \vec{w}_2, \vec{v}_2 \rangle \neq 0$.
Solving for $\vec{v}_2$ we get

$$\vec{v}_2 = \frac{\|\vec{v}_2\|^2}{\langle \vec{w}_2, \vec{v}_2 \rangle} \left[\vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1 \right]$$

We then take $\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{\|\vec{v}_1\|^2} \vec{v}_1$.

The Gram-Schmidt Procedure

We need to prove that a vector $\vec{v}_2$ defined this way does have the desired property.

Defining $\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{||\vec{v}_1||^2} \vec{v}_1$ gives

$$\langle \vec{v}_1, \vec{v}_2 \rangle = \left\langle \vec{v}_1, \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{||\vec{v}_1||^2} \vec{v}_1 \right\rangle = \langle \vec{v}_1, \vec{w}_2 \rangle - \langle \vec{w}_2, \vec{v}_1 \rangle = 0$$

Consequently, $\{ \vec{v}_1, \vec{v}_2 \}$ is an orthogonal set of 2 non-zero vectors in $\mathbb{W}$ and hence is an orthogonal basis for $\mathbb{W}$.

We can continue to repeat this procedure for 3, 4, etc. dimensional subspaces $\mathbb{W}$ of $\mathbb{V}$.
Doing so produces the following result.

The Gram-Schmidt Procedure

Theorem 9.3.1 - Gram-Schmidt Orthogonalization Theorem

Let $\{ \vec{w}_1, \dots, \vec{w}_n \}$ be a basis for an inner product space $\mathbb{W}$. If we define $\vec{v}_1, \dots, \vec{v}_n$ successively as follows:

$$\vec{v}_1 = \vec{w}_1$$

$$\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1 \rangle}{||\vec{v}_1||^2} \vec{v}_1$$

$$\vec{v}_i = \vec{w}_i - \frac{\langle \vec{w}_i, \vec{v}_1 \rangle}{||\vec{v}_1||^2} \vec{v}_1 - \frac{\langle \vec{w}_i, \vec{v}_2 \rangle}{||\vec{v}_2||^2} \vec{v}_2 - \cdots - \frac{\langle \vec{w}_i, \vec{v}_{i-1} \rangle}{||\vec{v}_{i-1}||^2} \vec{v}_{i-1} \qquad \text{for } 3 \leq i \leq n$$

then $\{ \vec{v}_1, \dots, \vec{v}_i \}$ is an orthogonal basis for $\mathrm{Span}\{ \vec{w}_1, \dots, \vec{w}_i \}$ for $1 \leq i \leq n$.

The algorithm defined in this theorem is called the Gram-Schmidt procedure.
This theorem shows that every finite dimensional inner product space has an orthonormal basis. We will use this fact constantly throughout the rest of the course.

The Gram-Schmidt Procedure

Example

Use the Gram-Schmidt procedure to find an orthonormal basis for the subspace of $\mathbb{R}^4$ defined by

$$\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$$

Solution

Label the vectors in the spanning set for $\mathbb{S}$ as $\vec{w}_1, \vec{w}_2, \vec{w}_3$ respectively.

First Step: We first take $\vec{v}_1 = \vec{w}_1 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$. Then, $\{\vec{v}_1\}$ is an orthogonal basis for $\operatorname{Span}\{\vec{w}_1\}$.

Second Step: $\vec{v}_2 = \vec{w}_2 - \frac{\langle \vec{w}_2, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 = \begin{bmatrix}-1/2\\1\\1\\1/2\end{bmatrix}$

For simplicity in the next calculations, we multiply this vector by 2 to get $\vec{v}_2 = \begin{bmatrix}-1\\2\\2\\1\end{bmatrix}$. Then, $\{\vec{v}_1, \vec{v}_2\}$ is an orthogonal basis for $\operatorname{Span}\{\vec{w}_1, \vec{w}_2\}$.

The Gram-Schmidt Procedure

Example

Use the Gram-Schmidt procedure to find an orthonormal basis for the subspace of $\mathbb{R}^4$ defined by

$$\mathbb{S} = \operatorname{Span}\left\{\begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$$

Solution

Third Step: $\vec{v}_3 = \vec{w}_3 - \frac{\langle \vec{w}_3, \vec{v}_1\rangle}{\|\vec{v}_1\|^2}\vec{v}_1 - \frac{\langle \vec{w}_3, \vec{v}_2\rangle}{\|\vec{v}_2\|^2}\vec{v}_2 = \begin{bmatrix}2/5\\1/5\\1/5\\-2/5\end{bmatrix}$ So, we take $\vec{v}_3 = \begin{bmatrix}2\\1\\1\\-2\end{bmatrix}$.

We now have that $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthogonal basis for $\mathbb{S}$. To obtain an orthonormal basis for $\mathbb{S}$, we simply divide each vector in this basis by its length. Thus, we find that an orthonormal basis for $\mathbb{S}$ is

$$\left\{\begin{bmatrix}1/\sqrt{2}\\0\\0\\1/\sqrt{2}\end{bmatrix}, \begin{bmatrix}-1/\sqrt{10}\\2/\sqrt{10}\\2/\sqrt{10}\\1/\sqrt{10}\end{bmatrix}, \begin{bmatrix}2/\sqrt{10}\\1/\sqrt{10}\\1/\sqrt{10}\\-2/\sqrt{10}\end{bmatrix}\right\}$$

The Gram-Schmidt Procedure

Example

Use the Gram-Schmidt procedure to find an orthogonal basis for the subspace $\mathbb{W}$ of $M_{2\times 2}(\mathbb{R})$ spanned by

$$\{A_1, A_2, A_3, A_4\} = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Solution

Take $B_1 = A_1$. Then

$$A_2 - \frac{\langle A_2, B_1 \rangle}{\|B_1\|^2} B_1 = \begin{bmatrix} -2 & -3 \\ 1 & -1 \end{bmatrix} - \frac{-10}{7} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -4/7 & -1/7 \\ -3/7 & 3/7 \end{bmatrix}$$

So, we take $B_2 = \begin{bmatrix} 4 & 1 \\ 3 & -3 \end{bmatrix}$. Next, we find that

$$A_3 - \frac{\langle A_3, B_1 \rangle}{\|B_1\|^2} B_1 - \frac{\langle A_3, B_2 \rangle}{\|B_2\|^2} B_2 = \begin{bmatrix} -3 & -1 \\ -2 & 2 \end{bmatrix} - \frac{-1}{7} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - \frac{-25}{35} \begin{bmatrix} 4 & 1 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

What happened? A_3 is a linear combination of B_1 and B_2. In particular,

$$A_3 = \frac{-1}{7} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} + \frac{-25}{35} \begin{bmatrix} 4 & 1 \\ 3 & -3 \end{bmatrix}$$

So, A_3 can be ignored and we can move to the next vector in the set.

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Example

Use the Gram-Schmidt procedure to find an orthogonal basis for the subspace $\mathbb{W}$ of $M_{2\times 2}(\mathbb{R})$ spanned by

$$\{A_1, A_2, A_3, A_4\} = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -2 & -3 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

Solution

We then have

$$\begin{aligned} B_3 &= A_4 - \frac{\langle A_4, B_1 \rangle}{\|B_1\|^2} B_1 - \frac{\langle A_4, B_2 \rangle}{\|B_2\|^2} B_2 \\ &= \begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} - \frac{7}{7} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} - \frac{28}{35} \begin{bmatrix} 4 & 1 \\ 3 & -3 \end{bmatrix} \\ &= \begin{bmatrix} 14/5 & -14/5 \\ -7/5 & 7/5 \end{bmatrix} \end{aligned}$$

Thus, $\{B_1, B_2, B_3\}$ is an orthogonal basis for $\mathbb{W}$.