The Rank-Nullity Theorem

In This Lecture

- We will extend the definitions of range and kernel to general linear mappings.
- We will prove the very important Rank-Nullity Theorem.

Subspaces of a Linear Mapping

Definition: Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. We define the kernel of L by

$$\mathrm{Ker}(L) = \{\vec{v} \in \mathbb{V} \mid L(\vec{v}) = \vec{0}_{\mathbb{W}}\}$$

Definition: Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. We define the range of L by

$$\mathrm{Range}(L) = \{L(\vec{v}) \mid \vec{v} \in \mathbb{V}\}$$

Subspaces of a Linear Mapping

Example

Find a basis for the range and kernel of the linear mapping $L : \mathbb{R}^3 \to P_2(\mathbb{R})$ defined by

$$L(a, b, c) = a + (a + b + c)x^2$$

Solution

First, observe that every vector in the range of L has the form

$$a + (a + b + c)x^2 = a + ax^2 + (b + c)x^2 = a(1 + x^2) + (b + c)x^2$$

Hence, the set $\{1 + x^2, x^2\}$ spans the range of L and is clearly linearly independent, so it is a basis for $\mathrm{Range}(L)$.

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathrm{Ker}(L)$. Then,

$$0 + 0x + 0x^2 = L(a, b, c) = a + (a + b + c)x^2$$

This implies that, $a = 0$ and $b = -c$.

Thus, every vector in $\mathrm{Ker}(L)$ has the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ -c \\ c \end{bmatrix} = c\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

Hence, $\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right\}$ is linearly independent and spans $\mathrm{Ker}(L)$, so it is a basis for $\mathrm{Ker}(L)$.

Subspaces of a Linear Mapping

Definition: Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. We define the rank of L by

$$\mathrm{rank}(L) = \dim(\mathrm{Range}(L))$$

We define the nullity of L to be

$$\mathrm{nullity}(L) = \dim(\mathrm{Ker}(L))$$

Theorem 8.2.1

Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. Then, $\mathrm{Ker}(L)$ is a subspace of $\mathbb{V}$ and $\mathrm{Range}(L)$ is a subspace of $\mathbb{W}$.

The Rank-Nullity Theorem

Theorem 8.2.2 - The Rank-Nullity Theorem

Let $\mathbb{V}$ be an n-dimensional vector space and let $\mathbb{W}$ be a vector space. If $L : \mathbb{V} \to \mathbb{W}$ is linear, then

$$\operatorname{rank}(L) + \operatorname{nullity}(L) = n$$

Proof

Assume that $\operatorname{nullity}(L) = k$. Then there exists a basis $\{\vec{v}_1, \cdots, \vec{v}_k\}$ for $\operatorname{Ker}(L)$. We can extend $\{\vec{v}_1, \cdots, \vec{v}_k\}$ to a basis $\{\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$ for $\mathbb{V}$.

We notice that $C = \{L(\vec{v}_{k+1}), \cdots, L(\vec{v}_n)\}$ is a set of $n-k$ vectors in the range of L. We need to prove this is a linearly independent spanning set for the range of L.

Consider

$$\begin{aligned} c_{k+1}L(\vec{v}_{k+1}) + \cdots + c_nL(\vec{v}_n) &= \vec{0} \\ L(c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n) &= \vec{0} \end{aligned}$$

This implies that $c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n \in \operatorname{Ker}(L)$.

Thus, there exist $d_1, \ldots, d_k \in \mathbb{R}$ such that

$$c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n = d_1\vec{v}_1 + \cdots + d_k\vec{v}_k$$

We then have

$$-d_1\vec{v}_1 - \cdots - d_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n = \vec{0}$$

Hence, $d_1 = \cdots = d_k = c_{k+1} = \cdots = c_n = 0$ since $\{\vec{v}_1, \cdots, \vec{v}_k, \vec{v}_{k+1}, \cdots, \vec{v}_n\}$ is linearly independent. Thus, the set C is linearly independent.

The Rank-Nullity Theorem

Theorem 8.2.2 - The Rank-Nullity Theorem

Let $\mathbb{V}$ be an n-dimensional vector space and let $\mathbb{W}$ be a vector space. If $L : \mathbb{V} \to \mathbb{W}$ is linear, then

$$\operatorname{rank}(L) + \operatorname{nullity}(L) = n$$

Proof (continued)

Let $\vec{y} \in \operatorname{Range}(L)$. Then, there exists $\vec{v} \in \mathbb{V}$ such that

$$\begin{aligned} \vec{y} &= L(\vec{v}) \\ &= L(c_1\vec{v}_1 + \cdots + c_k\vec{v}_k + c_{k+1}\vec{v}_{k+1} + \cdots + c_n\vec{v}_n) \\ &= c_1L(\vec{v}_1) + \cdots + c_kL(\vec{v}_k) + c_{k+1}L(\vec{v}_{k+1}) + \cdots + c_nL(\vec{v}_n) \\ &= \vec{0} + \cdots + \vec{0} + c_{k+1}L(\vec{v}_{k+1}) + \cdots + c_nL(\vec{v}_n) \\ &= c_{k+1}L(\vec{v}_{k+1}) + \cdots + c_nL(\vec{v}_n) \end{aligned}$$

Hence

$$\vec{y} \in \operatorname{Span}\{L(\vec{v}_{k+1}), \ldots, L(\vec{v}_n)\}$$

Therefore, we have shown that the set $C = \{L(\vec{v}_{k+1}), \ldots, L(\vec{v}_n)\}$ is a basis for $\operatorname{Range}(L)$ and hence

$$\operatorname{rank}(L) = n - k = n - \operatorname{nullity}(L)$$

□

The Rank-Nullity Theorem

Example

Let $\mathbb{U}$ be the subspace of $M_{2\times 2}(\mathbb{R})$ of upper triangular matrices and let $L : \mathbb{U} \to M_{2\times 2}(\mathbb{R})$ be the linear mapping defined by

$$L\left(\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}\right) = \begin{bmatrix} a-b & a+c \\ b+c & 0 \end{bmatrix}$$

Find the rank and nullity of L.

Solution

Every matrix $B \in \operatorname{Range}(L)$ has the form

$$B = \begin{bmatrix} a-b & a+c \\ b+c & 0 \end{bmatrix} = a\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Thus, $\operatorname{Range}(L) = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

Since $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ is clearly linearly independent, it is a basis for $\operatorname{Range}(L)$.

Thus, $\operatorname{rank}(L) = 2$ and hence the Rank-Nullity Theorem gives

$$\operatorname{nullity}(L) = \dim \mathbb{U} - \operatorname{rank}(L) = 3 - 2 = 1$$

The Rank-Nullity Theorem

Example

Let $L : P_2(\mathbb{R}) \to M_{2\times 3}(\mathbb{R})$ be the linear mapping defined by

$$L(a + bx + cx^2) = \begin{bmatrix} a & a & c \\ a & a & c \end{bmatrix}$$

Find the rank and nullity of L.

Solution

Let $a + bx + cx^2 \in \operatorname{Ker}(L)$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = L(a + bx + cx^2) = \begin{bmatrix} a & a & c \\ a & a & c \end{bmatrix}$$

Equating corresponding entries, this implies that $a = 0$ and $c = 0$.
Thus, every vector in the kernel has the form $a + bx + cx^2 = 0 + bx + 0x^2 = bx$.
The set $\{x\}$ is clearly linearly independent, hence, a basis for the kernel of L is $\{x\}$. So, $\operatorname{nullity}(L) = 1$.
Then, by the Rank-Nullity Theorem, we get that

$$\operatorname{rank}(L) = \dim P_2(\mathbb{R}) - \operatorname{nullity}(L) = 3 - 1 = 2$$