Unitary Diagonalization and Schur's Theorem

Last Lecture

We defined unitary matrices.

In This Lecture

We will try to mimic what we did in the real case with orthogonal diagonalization.

Unitary Diagonalization and Schur's Theorem

Definition: If A and B are matrices such that $A = U^*BU$, where U is a unitary matrix, then we say that A and B are unitarily similar.

Note: If A and B are unitarily similar, then they are similar and hence have the same determinant, rank, eigenvalues, and trace.

Definition: If A is unitarily similar to a diagonal matrix D, then we say that A is unitarily diagonalizable.

Unitary Diagonalization and Schur's Theorem

In the real case, a matrix A is orthogonally diagonalizable if and only if it is symmetric (that is if $A^T = A$). Therefore, we will look at the complex equivalent of a symmetric matrix.

Definition: A matrix $A \in M_{n\times n}(\mathbb{C})$ is called Hermitian if $A^* = A$.

Notes:

1. If A is Hermitian, then we have $\overline{(A)_{ij}} = A_{ji}$. Hence, the diagonal entries of A must be real, and for $i \neq j$ the ij-th entry must be the complex conjugate of the ji-th entry.
2. If $A \in M_{n\times n}(\mathbb{R})$ is symmetric, then A is also Hermitian.

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.1

An $n \times n$ matrix A is Hermitian if and only if

$$\langle \vec{z}, A\vec{w}\rangle = \langle A\vec{z}, \vec{w}\rangle$$

for all $\vec{z}, \vec{w} \in \mathbb{C}^n$.

Proof

If A is Hermitian, then we have $A^* = A$, so by Theorem 11.4.5 we get

$$\langle A\vec{z}, \vec{w}\rangle = \langle \vec{z}, A^*\vec{w}\rangle = \langle \vec{z}, A\vec{w}\rangle$$

If $\langle \vec{z}, A\vec{w}\rangle = \langle A\vec{z}, \vec{w}\rangle$, then we have

$$\begin{aligned} \vec{z}\cdot\overline{A\vec{w}} &= (A\vec{z})\cdot\overline{\vec{w}} \\ \vec{z}^T\,\overline{A}\,\overline{\vec{w}} &= \vec{z}^T A^T\,\overline{\vec{w}} \end{aligned}$$

Since this is valid for all $\vec{z}, \vec{w} \in \mathbb{C}^n$, we get by Theorem 3.1.4 from Linear Algebra 1 that $\overline{A} = A^T$. Taking the conjugate of both sides gives $A = A^*$ as required. □

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.2 – Schur's Theorem

If A is an $n \times n$ matrix, then A is unitarily similar to an upper triangular matrix whose diagonal entries are the eigenvalues of A.

The proof of this theorem is essentially the same as the proof of the triangularization theorem, except we don't have to worry anymore about the eigenvalues being real.
It is recommended that you read over the proof in the course notes and compare to the proof of the Triangularization Theorem.

Notes:

1. We often rearrange this to write $A = UTU^*$, which is called the Schur decomposition of A.
2. The fact that we can unitarily triangularize every matrix in $M_{n\times n}(\mathbb{C})$ is amazing and extermely useful. That is, Schur's Theorem is an extermely important theorem in linear algebra.

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.3 – Spectral Theorem for Hermitian Matrices

If A is Hermitian, then it is unitarily diagonalizable.

Of course, the proof of this theorem should be very similar to our proof of the principal axis theorem.
If necessary refer back to Lecture 19: Orthogonal Diagonalization, and look over the proof of Theorem 10.2.1 and the proof of the Principal Axis Theorem. Really try to think about the strategy being used.

In the proof of Theorem 10.2.1:

- We used the fact that A and D are similar to prove that A must have the same property as D, namely the property that $D^T = D$.

In the proof of the Principal Axis Theorem:

- We applied the Triangularization Theorem to get that A is orthogonally similar to an upper triangular matrix T.
- We then used the fact that A and T are similar to prove that T has the same property as A, namely $A^T = A$.
- We then proved that every upper triangular symmetric matrix is diagonal.

Therefore, to prove the Spectral Theorem for Hermitian Matrices, we will use the same strategy:

- We will use Schur's theorem to get that A is unitarily similar to an upper triangular matrix T.
- We will then use this fact to prove that T must have the same property as A. That is, T must also be Hermitian.
- We will prove that every upper triangular Hermitian matrix is in fact diagonal.

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.3 – Spectral Theorem for Hermitian Matrices

If A is Hermitian, then it is unitarily diagonalizable.

Proof

By Schur's Theorem there exists a unitary matrix U such that $U^*AU = T$ is upper triangular.
Since $A^* = A$, we get that

$$T^* = (U^*AU)^* = U^*A^*(U^*)^* = U^*AU = T$$

Hence, T is Hermitian. Since T^* is lower triangular, T is also lower triangular.
This implies that T must be diagonal. □

Unitary Diagonalization and Schur's Theorem

The other important result proven is that all eigenvalues of a Hermitian matrix are real.

We have shown that A is similar to the diagonal matrix T and therefore they have the same eigenvalues.
Since $T =$ is diagonal we can write $T = \text{diag}(t_{11}, \dots, t_{nn})$ and the eigenvalues of T are the entries t_{11} to t_{nn}.
Since T is also Hermitian we have that

$$\text{diag}(t_{11}, \dots, t_{nn}) = T = T^* = \text{diag}(\overline{t_{11}}, \dots, \overline{t_{nn}})$$

Thus, every eigenvalue of T and hence A satisfies $t_{ii} = \overline{t_{ii}}$ and hence are real.

Theorem 11.5.4

Every eigenvalue of a Hermitian matrix is real.

Note: Note that since every real symmetric matrix is Hermitian, I have also proven that all eigenvalues of a real symmetric matrix are real (as I promised I would back in lecture 19).

Unitary Diagonalization and Schur's Theorem

The converse of the Spectral Theorem for Hermitian matrices is not true.

That is, there are unitarily diagonalizable matrices which are not Hermitian.

Example

The matrix $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ is unitarily diagonalizable, but not Hermitian.

We notice that this matrix satisfies a condition that is very similar to being symmetric. That is, it satisfies $A^T = -A$. We call such a matrix skew-symmetric.

Definition: Let $A \in M_{n \times n}(\mathbb{C})$. If $A^* = -A$, then A is called skew-Hermitian.

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.5

Every skew-Hermitian matrix A is unitarily diagonalizable.

You should write down in detail what you think the *strategy* for this proof will be, and then compare your strategy with mine.

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.5

Every skew-Hermitian matrix A is unitarily diagonalizable.

The strategy I will use is:

- First use Schur's theorem to get that A is unitarily similar to an upper triangular matrix T.
- Then prove that T has the same property as A. That is, prove that T is also skew-Hermitian.
- Then, prove that every upper triangular skew-Hermitian matrix is diagonal.

Proof

If A is skew-Hermitian, then $A^* = -A$.
By Schur's Theorem, there exists a unitary matrix U and upper triangular matrix T such that $U^*AU = T$.
Then we have

$$T^* = (U^*AU)^* = U^*A^*U = U^*(-A)U = -(U^*AU) = -T$$

Hence, T is skew-Hermitian. Thus, $T = -T^*$ so T is both upper and lower triangular and hence diagonal. ☐

Unitary Diagonalization and Schur's Theorem

What does our proof say about the eigenvalues of a skew-Hermitian matrix?
We have

$$\text{diag}(t_{11}, \ldots, t_{nn}) = T = -T^* = \text{diag}(-\overline{t_{11}}, \ldots, -\overline{t_{nn}})$$

Thus, $t_{ii} = -\overline{t_{ii}}$, so all eigenvalues of a skew-Hermitian matrix are purely imaginary or 0.

Theorem 11.5.6

If λ is an eigenvalue of a skew-Hermitian matrix A, then $\lambda = ti$ for some $t \in \mathbb{R}$.

Unitary Diagonalization and Schur's Theorem

Example

The matrix $U = \begin{bmatrix} i/\sqrt{2} & 1/\sqrt{2} \\ -i/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ is unitarily diagonalizable but not Hermitian nor skew-Hermitian.

Notice that U is in fact unitary.

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.7

Every unitary matrix A is unitarily diagonalizable.

Write down explicitly what you think the strategy should be for this proof and compare your strategy with what I do.

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.7

Every unitary matrix A is unitarily diagonalizable.

The strategy I will use is:

- First use Schur's theorem to get that A is unitarily similar to an upper triangular matrix T.
- Then prove that T has the same property as A. That is, prove that T is also unitary.
- Then, prove that every upper triangular unitary matrix is diagonal.

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.7

Every unitary matrix A is unitarily diagonalizable.

Proof

If A is unitary, then $AA^* = I$.
By Schur's theorem, there exists a unitary matrix U and upper triangular matrix T such that $U^*AU = T$.
Notice that since U^*, A, and U are all unitary, then T is also unitary by Theorem 11.4.5 (3).

We now need to show that every upper triangular unitary matrix is diagonal.

Let $T = \begin{bmatrix} t_{11} & t_{12} & \cdots & t_{1n} \\ 0 & t_{22} & \cdots & t_{2n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & t_{nn} \end{bmatrix}$.
Since T is unitary, the columns of T form an orthonormal basis for $\mathbb{C}^n$. Hence, the first column must have length 1.
That is, we have

$$1 = \left\| \begin{bmatrix} t_{11} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\| = |t_{11}|$$

Unitary Diagonalization and Schur's Theorem

Theorem 11.5.7

Every unitary matrix A is unitarily diagonalizable.

Proof

Also, the columns are orthogonal, so we have

$$0 = \left\langle \begin{bmatrix} t_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} t_{12} \\ t_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\rangle = t_{11}\overline{t_{12}}$$

Since $|t_{11}| = 1$ we know that $t_{11} \neq 0$. Therefore, we must have $t_{12} = 0$.

Thus, the second column has the form $\begin{bmatrix} 0 \\ t_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

So, repeating what we did above but now using the second and third columns we find that $|t_{22}| = 1$ and $t_{13} = t_{23} = 0$.
Continuing in this way we get that T is diagonal. □

Note: This is not a formal proof. Instead of just saying "continuing in this way", we should use induction.

Unitary Diagonalization and Schur's Theorem

What have we proven about the eigenvalues of a unitary matrix?

Theorem 11.5.8

If λ is an eigenvalue of a unitary matrix A, then $|\lambda| = 1$.

Note: This means that λ can be any complex number on the unit circle in the complex plane.

Unitary Diagonalization and Schur's Theorem

So, all Hermitian, skew-Hermitian, and unitary matrices are unitarily diagonalizable. Are there more? Yes!

The matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is also unitarily diagonalizable, but it is not Hermitian, skew-Hermitian, nor unitary.

This strategy of guessing and checking which class of matrices is unitarily diagonalizable is obviously not a good approach.

What approach would be better? Try to think about this before the next lecture.