Eigenvalues and Eigenvectors

Last Lecture

- We looked at how to find the matrix of a linear operator with respect to a basis $\mathcal{B}$.
- We stated our goal of determining if a linear operator L has a basis $\mathcal{B}$ such that the matrix of L with respect to $\mathcal{B}$ is diagonal, and if it does, how to find $\mathcal{B}$.

In This Lecture

- We will figure out how to meet our goal.

Eigenvalues and Eigenvectors

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator.
Assume that there is a basis $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ such that the matrix of L with respect to $\mathcal{B}$ is diagonal.
Let $[L]_\mathcal{B} = \text{diag}(\lambda_1, \dots, \lambda_n)$.
In the last lecture we found that $[L]_\mathcal{B}$ was similar to the standard matrix $A = [L]$ of L.
In particular, using the change of coordinates matrix $P = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$ we got

$$\begin{aligned} P^{-1}AP &= \text{diag}(\lambda_1, \dots, \lambda_n) \\ AP &= P\,\text{diag}(\lambda_1, \dots, \lambda_n) \\ A\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} &= \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}\text{diag}(\lambda_1, \dots, \lambda_n) \\ \begin{bmatrix} A\vec{v}_1 & \cdots & A\vec{v}_n \end{bmatrix} &= \begin{bmatrix} \lambda_1\vec{v}_1 & \cdots & \lambda_n\vec{v}_n \end{bmatrix} \end{aligned}$$

Comparing columns we see that we must have $A\vec{v}_i = \lambda_i\vec{v}_i$ for $1 \leq i \leq n$.
Moreover, since P is invertible, the columns of P must be linearly independent.
In particular, $\vec{v}_i \neq \vec{0}$ for $1 \leq i \leq n$.

On the other hand, if $A\vec{v}_i = \lambda_i\vec{v}_i$ for $1 \leq i \leq n$, then we have

$$[L(\vec{v}_i)]_\mathcal{B} = [\lambda_i\vec{v}_i]_\mathcal{B} = \lambda_i\vec{e}_i, \qquad 1 \leq i \leq n$$

Hence,

$$[L]_\mathcal{B} = \begin{bmatrix} \lambda_1\vec{e}_1 & \cdots & \lambda_n\vec{e}_n \end{bmatrix} = \text{diag}(\lambda_1, \dots, \lambda_n)$$

So, $[L]_\mathcal{B}$ is diagonal.

Eigenvalues and Eigenvectors

Definition: Let A be an $n \times n$ matrix. If there is a non-zero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$ for some scalar λ, then λ is called an eigenvalue of A and $\vec{v}$ is called an eigenvector of A corresponding to λ. We call $(\lambda, \vec{v})$ an eigenpair.

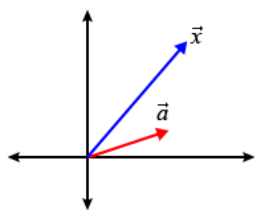
Definition: Let L be a linear operator on $\mathbb{R}^n$. If there is a non-zero vector $\vec{v}$ such that $L(\vec{v}) = \lambda\vec{v}$ for some scalar λ, then λ is called an eigenvalue of L and $\vec{v}$ is called an eigenvector of L corresponding to λ.

Eigenvalues and Eigenvectors

Example 1

Let $\vec{a} \in \mathbb{R}^n$, $\vec{a} \neq \vec{0}$. Thinking geometrically, what are the eigenvectors and corresponding eigenvalues of $\text{proj}_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^n$?

Solution



We want to find all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $\text{proj}_{\vec{a}}\vec{x} = \lambda\vec{x}$.

Eigenvalues and Eigenvectors

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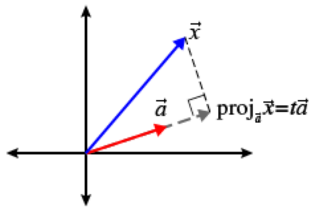

We want to find all vectors $\vec{x}$ in $\mathbb{R}^n$ such that $\text{proj}_{\vec{a}}\vec{x} = \lambda\vec{x}$.
By definition, we know that $\text{proj}_{\vec{a}}\vec{x}$ is a scalar multiple of $\vec{a}$.
Hence, any **non-zero** scalar multiple of $\vec{a}$ should be an eigenvector.
Indeed we have

$$\text{proj}_{\vec{a}}(t\vec{a}) = \frac{(t\vec{a}) \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = t\vec{a} = (1)(t\vec{a})$$

Thus, $t\vec{a}$ for $t \neq 0$ is an eigenvector with eigenvalue $\lambda = 1$.

Eigenvalues and Eigenvectors

Example 1

Let $\vec{a} \in \mathbb{R}^n$, $\vec{a} \neq \vec{0}$. Thinking geometrically, what are the eigenvectors and corresponding eigenvalues of $\text{proj}_{\vec{a}} : \mathbb{R}^n \to \mathbb{R}^n$?

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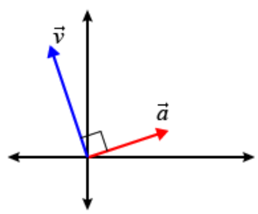


We observe that if $\vec{v}$ is any vector orthogonal to $\vec{a}$, then we have

$$\text{proj}_{\vec{a}}\vec{v} = \frac{\vec{v} \cdot \vec{a}}{\|\vec{a}\|^2}\vec{a} = \vec{0} = 0\vec{a}$$

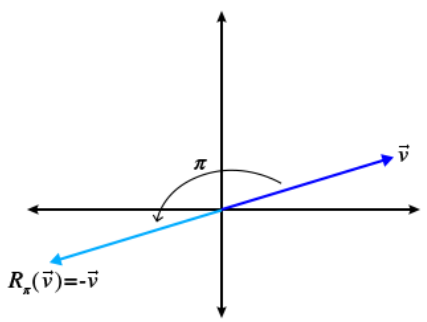
Hence, all **non-zero** vectors orthogonal to $\vec{a}$ are eigenvectors with corresponding eigenvalue 0.

Eigenvalues and Eigenvectors

Example 2

Thinking geometrically, what are the eigenvectors and corresponding eigenvalues of a rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by an angle θ where $0 < \theta < 2\pi$ radians?

Solution



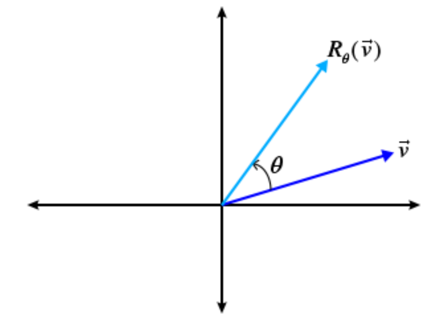
We first realize that if $\theta = \pi$ radians, then we are rotating $\vec{v}$ around to $-\vec{v}$.
Hence, every non-zero vector $\vec{v}$ is an eigenvector of R_π with corresponding eigenvalue -1.

Eigenvalues and Eigenvectors

Example 2

Thinking geometrically, what are the eigenvectors and corresponding eigenvalues of a rotation $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ by an angle θ where $0 < \theta < 2\pi$ radians?

Solution



On the other hand, if we are rotating by an angle θ with $0 < \theta < 2\pi$ and $\theta \neq \pi$, then the result will not be a scalar multiple of $\vec{v}$.
So, R_θ will not have any real eigenvalues.

Eigenvalues and Eigenvectors

Example 3

Consider $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$. Determine which of the following vectors are eigenvectors of A.

(a) $\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Solution

We have

$$A\vec{v}_1 = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -2 \end{bmatrix} = -2\vec{v}_1$$

Thus, $\vec{v}_1$ is an eigenvector of A with corresponding eigenvalue $\lambda_1 = -2$.

Eigenvalues and Eigenvectors

Example 3

Consider $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$. Determine which of the following vectors are eigenvectors of A.

(b) $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution

We have

$$A\vec{v}_2 = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 16 \\ 13 \\ 16 \end{bmatrix}$$

which is not a scalar multiple of $\vec{v}_2$.
Thus, $\vec{v}_2$ is not an eigenvector.

Eigenvalues and Eigenvectors

Example 3

Consider $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$. Determine which of the following vectors are eigenvectors of A.

(c) $\vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Solution

We have $A\vec{v}_3 = \vec{0} = 1\vec{v}_3$, so $\vec{v}_3$ is NOT an eigenvector because by definition the zero vector is not allowed to be an eigenvector!

Eigenvalues and Eigenvectors

Example 4

Consider $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$. Is $\lambda = 1$ an eigenvalue of A?

Solution

We need to determine if is there a non-zero vector $\vec{v}$ such that $A\vec{v} = 1\vec{v}$.
For this we would need

$$\begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

This gives

$$\begin{aligned} 3v_1 + 6v_2 + 7v_3 &= v_1 \\ 3v_1 + 3v_2 + 7v_3 &= v_2 \\ 5v_1 + 6v_2 + 5v_3 &= v_3 \end{aligned}$$

Moving the variables on the right side to the left side gives the homogeneous system

$$\begin{aligned} 2v_1 + 6v_2 + 7v_3 &= 0 \\ 3v_1 + 2v_2 + 7v_3 &= 0 \\ 5v_1 + 6v_2 + 4v_3 &= 0 \end{aligned}$$

Solving this system we find that the only solution is the trivial solution.
Hence, the only vector that satisfies $A\vec{v} = \vec{v}$ is the zero vector.
Hence, $\lambda = 1$ is not an eigenvalue of A.

Eigenvalues and Eigenvectors

Example 4

Consider $A = \begin{bmatrix} 3 & 6 & 7 \\ 3 & 3 & 7 \\ 5 & 6 & 5 \end{bmatrix}$. Is $\lambda = 1$ an eigenvalue of A?

Solution

It is instructive to notice that the coefficient matrix of the homogeneous system was

$$\begin{bmatrix} 2 & 6 & 7 \\ 3 & 2 & 7 \\ 5 & 6 & 4 \end{bmatrix} = \begin{bmatrix} 3-\lambda & 6 & 7 \\ 3 & 3-\lambda & 7 \\ 5 & 6 & 5-\lambda \end{bmatrix} = A - \lambda I$$