Bases for Row(A), Col(A), and Null(A)

Recall that the basis for a subspace S is a set of vectors that both spans S and is linearly independent.
We saw previously that every basis for a given subspace S will have the same number of vectors, and this number is known as the dimension of the subspace.

We will learn how to easily find bases for the special subspaces $\text{Row}(A)$, $\text{Col}(A)$, and $\text{Null}(A)$.

The Basis of the Rowspace of a Matrix

Theorem 3.4.5

Let B be the RREF of an $m \times n$ matrix A. Then the non-zero rows of B form a basis for $\text{Row}(A)$, and hence the dimension of $\text{Row}(A)$ equals the rank of A.

Example

To find a basis for the rowspace of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$, we first need to find the RREF of A:

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \\ R_4 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in RREF, and so its non-zero rows form a basis for the rowspace of A.

Thus, $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for the rowspace of A.

The Basis of the Rowspace of a Matrix

Theorem 3.4.5

Let B be the RREF of an $m \times n$ matrix A. Then the non-zero rows of B form a basis for $\mathrm{Row}(A)$, and hence the dimension of $\mathrm{Row}(A)$ equals the rank of A.

Proof

Theorem 3.4.4 tells us that $\mathrm{Row}(B) = \mathrm{Row}(A)$, so we know that the non-zero rows of B form a spanning set for $\mathrm{Row}(A)$.
All we need to do now is show that they are linearly independent.

The Basis of the Rowspace of a Matrix

Theorem 3.4.5

Let B be the RREF of an $m \times n$ matrix A. Then the non-zero rows of B form a basis for $\mathrm{Row}(A)$, and hence the dimension of $\mathrm{Row}(A)$ equals the rank of A.

Proof

So, let $\vec{b}_1^T, \ldots, \vec{b}_r^T$ be the non-zero rows of B, and let's look at the vector equation

$$t_1\vec{b}_1 + \cdots + t_r\vec{b}_r = \vec{0}$$

Now, as each of the $\vec{b}_i$ are non-zero, and because B is in RREF, each $\vec{b}_i$ has a leading 1. $(1 \le i \le r)$
Suppose the leading 1 for $\vec{b}_i$ is in its j-th component. $(1 \le j \le n)$
Since B is in **reduced** row echelon form, we know that the other rows have a 0 for their j-th component.
So, let's look at the j-th component of our vector equation:

$$t_1(\vec{b}_1)_j + \cdots + t_i(\vec{b}_i)_j + \cdots + t_r(\vec{b}_r)_j = 0$$

But since we now know that $(\vec{b}_i)_j = 1$, and $(\vec{b}_k)_j = 0$ for $k \neq i$ $(1 \le k \le r)$, our equation becomes

$$0 + \cdots + t_i(1) + \cdots + 0 = 0 \Rightarrow t_i = 0$$

This result holds for all $i = 1, \ldots, r$. Thus, we have shown that the only solution to the vector equation

$$t_1\vec{b}_1 + \cdots + t_r\vec{b}_r = \vec{0}$$

is $t_1 = \cdots = t_r = 0$.
By definition, this means that the vectors $\vec{b}_1, \ldots, \vec{b}_r$ are linearly independent.
As such, it follows that the non-zero rows of B form a basis for $\mathrm{Row}(A)$. □

The Basis of the Columnspace of a Matrix

A basis for the columnspace of a matrix A is not simply just the non-zero columns of the RREF of A.

Theorem 3.4.6

Suppose that B is the RREF of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis for $\text{Col}(A)$. Hence, the dimension of $\text{Col}(A)$ equals $\text{rank}(A)$.

The Basis of the Columnspace of a Matrix

Theorem 3.4.6

Suppose that B is the RREF of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis for $\text{Col}(A)$. Hence, the dimension of $\text{Col}(A)$ equals $\text{rank}(A)$.

Example

Let's find a basis for the columnspace of $A = \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 8 \\ 3 & 1 & 2 & 9 & -3 \\ -2 & -1 & -2 & -6 & 1 \end{bmatrix}$.

First, we need its RREF.

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 8 \\ 3 & 1 & 2 & 9 & -3 \\ -2 & -1 & -2 & -6 & 1 \end{bmatrix} \begin{matrix} \\ \frac{1}{2}R_2 \\ R_3 - 3R_1 \\ R_4 + 2R_1 \end{matrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & -2 & -4 & 0 & -6 \\ 0 & 1 & 2 & 0 & 3 \end{bmatrix} \begin{matrix} \\ \\ R_3 + 2R_2 \\ R_4 - R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{2}R_3 \sim \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_1 - R_3 \\ R_2 - 4R_3 \\ \\ R_4 - R_3 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The Basis of the Columnspace of a Matrix

Theorem 3.4.6

Suppose that B is the RREF of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis for $\mathrm{Col}(A)$. Hence, the dimension of $\mathrm{Col}(A)$ equals $\mathrm{rank}(A)$.

Example

Now we see that the RREF of $A = \begin{bmatrix} 1 & 1 & 2 & 3 & 1 \\ 0 & 2 & 4 & 0 & 8 \\ 3 & 1 & 2 & 9 & -3 \\ -2 & -1 & -2 & -6 & 1 \end{bmatrix}$ is the matrix $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The RREF of A has a leading 1 in the first, second, and fifth columns.Thus, the first, second, and fifth columns of A form a basis for $\mathrm{Col}(A)$.

That is, $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 8 \\ -3 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathrm{Col}(A)$.

The Basis of the Columnspace of a Matrix

Theorem 3.4.6

Suppose that B is the RREF of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis for $\mathrm{Col}(A)$. Hence, the dimension of $\mathrm{Col}(A)$ equals $\mathrm{rank}(A)$.

Proof

There are two facts that are going to be very important in this proof:

- For any two row equivalent matrices C and D, we have that $C\vec{x} = \vec{0}$ if and only if $D\vec{x} = \vec{0}$.
- We can remove elements of a set that can be written as a linear combination of the other elements of the set, without changing the span of the set (Theorem 1.2.3).

The Basis of the Columnspace of a Matrix

Theorem 3.4.6

Suppose that B is the RREF of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis for $\mathbf{Col}(A)$. Hence, the dimension of $\mathbf{Col}(A)$ equals $\mathbf{rank}(A)$.

Proof

Let $A = [\vec{a}_1 \quad \cdots \quad \vec{a}_n]$ and let the RREF of A be the matrix $B = [\vec{b}_1 \quad \cdots \quad \vec{b}_n]$.
Let's say that $\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}$ are the columns of B with leading 1s, and that $\vec{b}_{j_1}, \ldots, \vec{b}_{j_{n-k}}$ are the columns that do not contain leading 1s.
Notice that because B is in RREF, the vectors $\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}$ are the first k standard basis vectors of $\mathbb{R}^m$. Therefore, we know they are linearly independent.
Now let $B^* = [\vec{b}_{i_1} \quad \cdots \quad \vec{b}_{i_k}]$, and let $A^* = [\vec{a}_{i_1} \quad \cdots \quad \vec{a}_{i_k}]$ be the matrix made up of the columns of A that correspond to the columns of B that contain a leading 1.
Then A^* is row equivalent to B^*, and we know that $A^*\vec{x} = \vec{0}$ if and only if $B^*\vec{x} = \vec{0}$.
Since the columns of B^* are linearly independent, we know that the only solution to $B^*\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
Therefore, the only solution to $A^*\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$, which means that the columns of A^* are linearly independent.
So, we have shown that the columns of A that correspond to columns of B with leading 1s form a linearly independent set.

The Basis of the Columnspace of a Matrix

Theorem 3.4.6

Suppose that B is the RREF of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis for $\mathrm{Col}(A)$. Hence, the dimension of $\mathrm{Col}(A)$ equals $\mathrm{rank}(A)$.

Proof

Next, we need show that these columns, namely the columns of A^*, are a spanning set for $\mathrm{Col}(A)$.
Remember that the columns of A span $\mathrm{Col}(A)$ by definition. Now we just need to show that we can remove the columns of A that are not columns of A^*.
Let's now consider a vector $\vec{b}_{j_l}$, that is a column of B that does not have a leading 1. $(1 \leq l \leq n-k)$
Then its only non-zero entries occur in components where columns of B to the left of $\vec{b}_{j_l}$ have leading 1s.
So, we know that $\vec{b}_{j_l}$ can be written as a linear combination of the columns of B with leading 1s strictly to its left.
But we may say that $\vec{b}_{j_l}$ can be written as a linear combination of the standard basis vectors for $\mathbb{R}^m$, $\vec{b}_{i_1}, \ldots, \vec{b}_{i_k}$, regardless of its position in B.
Thus, if we let $\vec{b}_{j_l} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix}$ for $c_1, \ldots, c_m \in \mathbb{R}$ which are zero or non-zero as determined by the matrix B and the position of $\vec{b}_{j_l}$, we have

$$\vec{b}_{j_l} = c_1\vec{b}_{i_1} + \cdots + c_k\vec{b}_{i_k} \Leftrightarrow c_1\vec{b}_{i_1} + \cdots + c_k\vec{b}_{i_k} - \vec{b}_{j_l} = \vec{0} \Leftrightarrow \begin{bmatrix} \vec{b}_{i_1} & \cdots & \vec{b}_{i_k} & \vec{b}_{j_l} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \\ -1 \end{bmatrix} = \vec{0}$$

The Basis of the Columnspace of a Matrix

Theorem 3.4.6

Suppose that B is the RREF of A. Then the columns of A that correspond to the columns of B with leading 1s form a basis for $\mathrm{Col}(A)$. Hence, the dimension of $\mathrm{Col}(A)$ equals $\mathrm{rank}(A)$.

Proof

But the corresponding matrix $\begin{bmatrix} \vec{a}_{i_1} & \cdots & \vec{a}_{i_k} & \vec{a}_{j_l} \end{bmatrix}$ is row equivalent to $\begin{bmatrix} \vec{b}_{i_1} & \cdots & \vec{b}_{i_k} & \vec{b}_{j_l} \end{bmatrix}$.

So we have that $\begin{bmatrix} \vec{a}_{i_1} & \cdots & \vec{a}_{i_k} & \vec{a}_{j_l} \end{bmatrix}\begin{bmatrix} c_1 \\ \vdots \\ c_k \\ -1 \end{bmatrix} = \vec{0}$.

Or more importantly that $\vec{a}_{j_l} = c_1\vec{a}_{i_1} + \cdots + c_k\vec{a}_{i_k}$.

Hence, we see that $\vec{a}_{j_l}$ can be written as a linear combination of the columns of A^*, $\vec{a}_{i_1}, \dots, \vec{a}_{i_k}$.
Since this holds for l such that $1 \leq l \leq n-k$, the vectors $\vec{a}_{j_1}, \dots, \vec{a}_{j_{n-k}}$ can each be written as a linear combination of the vectors $\vec{a}_{i_1}, \dots, \vec{a}_{i_k}$.
As such, we have that $\mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \mathrm{Span}\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}, \vec{a}_{j_1}, \dots, \vec{a}_{j_{n-k}}\} = \mathrm{Span}\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}\}$.

Thus, we have shown that:

- $\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}\}$ is linearly independent.
- $\mathrm{Span}\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}\} = \mathrm{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = \mathrm{Col}(A)$.

Therefore, $\{\vec{a}_{i_1}, \dots, \vec{a}_{i_k}\}$ is a basis for $\mathrm{Col}(A)$. □

The Basis of the Nullspace of a Matrix

Ealier in the course, we saw that if a matrix A had rank r, then the general solution of $A\vec{x} = \vec{0}$ was expressed as the spanning set of $n-r$ vectors.
We will soon show that these vectors are linearly independent, and therefore form a basis for the nullspace of A.

Definition: Let A be an $m \times n$ matrix. We call the dimension of $\mathrm{Null}(A)$ the nullity of A and denote it by $\mathrm{nullity}(A)$.

The Basis of the Nullspace of a Matrix

Theorem 3.4.7

Let A be an $m \times n$ matrix with $\operatorname{rank}(A) = r$. Then the spanning set for the general solution of the homogeneous system $A\vec{x} = \vec{0}$ obtained by the method in Module 2 is a basis for $\operatorname{Null}(A)$, and the nullity of A is $n - r$.

Example

Let's find a basis for the nullspace of $A = \begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ -3 & 4 & -9 & -3 & 1 \\ -1 & 0 & -7 & -1 & -5 \end{bmatrix}$.

We need to find the general solution to $A\vec{x} = \vec{0}$, and so we begin by finding the RREF of A:

$$\begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ -3 & 4 & -9 & -3 & 1 \\ -1 & 0 & -7 & -1 & -5 \end{bmatrix} \begin{matrix} \\ R_2 + 3R_1 \\ R_3 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & -1 & -3 & 1 & -5 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 4 & -4 \end{bmatrix} \begin{matrix} \\ \\ \frac{1}{4}R_3 \end{matrix} \sim \begin{bmatrix} 1 & -1 & 4 & 2 & 0 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 - 2R_3 \\ R_2 - 3R_3 \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & -1 & 4 & 0 & 2 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{matrix} R_1 + R_2 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 7 & 0 & 6 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

The Basis of the Nullspace of a Matrix

Theorem 3.4.7

Let A be an $m \times n$ matrix with $\operatorname{rank}(A) = r$. Then the spanning set for the general solution of the homogeneous system $A\vec{x} = \vec{0}$ obtained by the method in Module 2 is a basis for $\operatorname{Null}(A)$, and the nullity of A is $n - r$.

Example

We now know that the RREF of A is

$$\begin{bmatrix} 1 & 0 & 7 & 0 & 6 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Thus we can see that the system $A\vec{x} = \vec{0}$ is equivalent to

$$\begin{array}{ccccccc} x_1 & & +7x_3 & & +6x_5 & = 0 \\ & x_2 & +3x_3 & & +4x_5 & = 0 \\ & & & x_4 & -x_5 & = 0 \end{array}$$

Replacing the variables x_3 and x_5 with the parameters s and t respectively, we get

$$\begin{array}{ccccccc} x_1 & & +7s & & +6t & = 0 \\ & x_2 & +3s & & 4t & = 0 \\ & & & x_4 & -t & = 0 \end{array}$$

The Basis of the Nullspace of a Matrix

Theorem 3.4.7

Let A be an $m \times n$ matrix with $\operatorname{rank}(A) = r$. Then the spanning set for the general solution of the homogeneous system $A\vec{x} = \vec{0}$ obtained by the method in Module 2 is a basis for $\operatorname{Null}(A)$, and the nullity of A is $n - r$.

Example

We now know that the RREF of A is

$$\begin{bmatrix} 1 & 0 & 7 & 0 & 6 \\ 0 & 1 & 3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

So we see that the general solution to $A\vec{x} = \vec{0}$ is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -7s - 6t \\ -3s - 4t \\ s \\ t \\ t \end{bmatrix} = s\begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -6 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus, we have that $\left\{ \begin{bmatrix} -7 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ -4 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\operatorname{Null}(A)$. Notice that $\operatorname{nullity}(A) = n - r = 5 - 3 = 2$.

The Basis of the Nullspace of a Matrix

Theorem 3.4.7

Let A be an $m \times n$ matrix with $\operatorname{rank}(A) = r$. Then the spanning set for the general solution of the homogeneous system $A\vec{x} = \vec{0}$ obtained by the method in Module 2 is a basis for $\operatorname{Null}(A)$, and the nullity of A is $n - r$.

Proof

Let $\{\vec{v}_1, \dots, \vec{v}_{n-r}\}$ be a spanning set for the general solution of $A\vec{x} = \vec{0}$ obtained from the RREF of A.
Consider the equation

$$t_1\vec{v}_1 + \cdots + t_{n-r}\vec{v}_{n-r} = \vec{0}$$

Looking at a general term $t_i\vec{v}_i$, suppose that $\vec{v}_i$ is the vector affiliated with the parameter s_i, which came from the variable x_j as shown below.

$$\vec{x} = s_1\vec{v}_1 + \cdots + s_i\vec{v}_i + \cdots + s_{n-r}\vec{v}_{n-r}$$

Then the j-th component of $\vec{v}_i$ is a 1. But, more importantly, the j-th component of all the other vectors in the set $\{\vec{v}_1, \dots, \vec{v}_{n-r}\}$ will be a 0.

The Basis of the Nullspace of a Matrix

Theorem 3.4.7

Let A be an $m \times n$ matrix with $\text{rank}(A) = r$. Then the spanning set for the general solution of the homogeneous system $A\vec{x} = \vec{0}$ obtained by the method in Module 2 is a basis for $\text{Null}(A)$, and the nullity of A is $n - r$.

Proof

So, if we look at the j-th scalar equation of the system

$$t_1\vec{v}_1 + \cdots + t_i\vec{v}_i + \cdots + t_{n-r}\vec{v}_{n-r} = \vec{0}$$

then we have

$$t_1(\vec{v}_1)_j + \cdots + t_i(\vec{v}_i)_j + \cdots + t_{n-r}(\vec{v}_{n-r})_j = 0 \Rightarrow 0 + \cdots + t_i(1) + \cdots + 0 = 0 \Rightarrow t_i = 0$$

But since this is true for any term $t_i\vec{v}_i$, we have shown that $t_1 = \cdots = t_{n-r} = 0$.
This is the only solution to our vector equation. And thus, the set $\{\vec{v}_1, \ldots, \vec{v}_{n-r}\}$ is linearly independent.
Since we already knew that it spanned the nullspace, we have shown that $\{\vec{v}_1, \ldots, \vec{v}_{n-r}\}$ is a basis for the nullspace. □