Block Multiplication

Previously, we looked at making a matrix by lining up the products $A\vec{e}_1\ A\vec{e}_2 \cdots A\vec{e}_n$.
This is an example of defining a matrix product (in this case, AI) by block multiplication.
Instead of thinking of matrix multiplication as one giant matrix times another giant matrix, each matrix can be broken into small blocks, and we can do our calculations with these small blocks.

Let A be an $m \times n$ matrix, and let B be an $n \times p$ matrix.
Let $\vec{b}_i$ be the i-th column of B, so that $B = [\vec{b}_1\ \vec{b}_2\ \cdots\ \vec{b}_p]$.
Then $AB = [A\vec{b}_1\ A\vec{b}_2\ \cdots\ A\vec{b}_p]$.

Block Multiplication

Example

Let $A = \begin{bmatrix} 1 & -3 \\ 0 & 4 \\ 2 & -1 \end{bmatrix}$.

Let $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$, and $\vec{z} = \begin{bmatrix} 7 \\ 0 \end{bmatrix}$.

$$A\vec{w} = \begin{bmatrix} (1)(1) + (-3)(2) \\ (0)(1) + (4)(2) \\ (2)(1) + (-1)(2) \end{bmatrix} = \begin{bmatrix} -5 \\ 8 \\ 0 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} (1)(0) + (-3)(-1) \\ (0)(0) + (4)(-1) \\ (2)(0) + (-1)(-1) \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$$

$$A\vec{y} = \begin{bmatrix} (1)(3) + (-3)(8) \\ (0)(3) + (4)(8) \\ (2)(3) + (-1)(8) \end{bmatrix} = \begin{bmatrix} -21 \\ 32 \\ -2 \end{bmatrix}$$

$$A\vec{z} = \begin{bmatrix} (1)(7) + (-3)(0) \\ (0)(7) + (4)(0) \\ (2)(7) + (-1)(0) \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 14 \end{bmatrix}$$

From this, we see that $A\begin{bmatrix} 1 & 0 & 3 & 7 \\ 2 & -1 & 8 & 0 \end{bmatrix} = \begin{bmatrix} A\vec{w} & A\vec{x} & A\vec{y} & A\vec{z} \end{bmatrix} = \begin{bmatrix} -5 & 3 & -21 & 7 \\ 8 & -4 & 32 & 0 \\ 0 & 1 & -2 & 14 \end{bmatrix}$

Block Multiplication

Another common reason to use block multiplication is if you have matrices with lots of zeros.

If you can partition the matrices so that you have blocks of all zeros, things can go a bit quicker.

Note: Let A be an $m \times n$ matrix. Then $O_{p,m}A = O_{p,n}$ and $AO_{n,p} = O_{m,p}$ for any p.

Block Multiplication

Example

Calculate $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 0 & -9 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 & 0 \\ -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

Solution

Instead of just diving in, we can instead partition our matrices as follows:

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 3 \\ 4 & 0 & -9 & 8 \end{array}\right]\left[\begin{array}{cc|cc} 1 & 2 & 0 & 0 \\ -9 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5 \end{array}\right]$$

So we let $A_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$, $A_{12} = \begin{bmatrix} 1 & 3 \\ -9 & 8 \end{bmatrix}$, $B_{11} = \begin{bmatrix} 1 & 2 \\ -9 & 0 \end{bmatrix}$, $B_{12} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = B_{21}$, and $B_{22} = \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}$.

Then we have

$$AB = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{21} + A_{12}B_{22} \end{bmatrix}$$

But since B_{12} and B_{21} are the zero matrix, we have $A_{12}B_{21} = O_{2,2}$ and $A_{11}B_{21} = O_{2,2}$.

Block Multiplication

Example

Calculate $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 4 & 0 & -9 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 0 \\ -9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$.

Solution

Thus, $AB = \begin{bmatrix} A_{11}B_{11} & A_{12}B_{22} \end{bmatrix}$.

$$A_{11}B_{11} = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -9 & 0 \end{bmatrix} = \begin{bmatrix} -17 & 2 \\ 4 & 8 \end{bmatrix}$$

$$A_{12}B_{22} = \begin{bmatrix} 1 & 3 \\ -9 & 8 \end{bmatrix} \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 18 \\ 0 & 13 \end{bmatrix}$$

So $AB = \begin{bmatrix} -17 & 2 & 0 & 18 \\ 4 & 8 & 0 & 13 \end{bmatrix}$.