Cramer's Rule

The cofactor method gives us the useful formula

$$A^{-1} = \frac{1}{\det A}(\operatorname{cof} A)^T$$

We can now apply this formula to a system of equations because, if A is invertible, we know that the solution to the system $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.
But this means that

$$\vec{x} = \frac{1}{\det A}(\operatorname{cof} A)^T\vec{b}$$

Recalling that matrix multiplication is the dot product of a row with a column, and that the rows of $(\operatorname{cof} A)^T$ are the columns of $\operatorname{cof} A$, then we see that

$$x_i = \frac{1}{\det A}(b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni})$$

The equation $b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni}$ looks like the calculation for a determinant expanded along the i-th column.
In fact, this is the determinant of the matrix N_i that can be obtained from A by replacing the i-th column with $\vec{b}$, since the cofactors of column i do not involve column i, and we would have $a_{ki} = b_k$ for all k $(1 \le k \le n)$.
Thus, we have

$$x_i = \frac{\det N_i}{\det A}$$

This expression for the solution to a system of equations is known as Cramer's Rule.

Cramer's Rule

Example

Use Cramer's Rule to solve this system of equations:

$$\begin{array}{rrcr} 4x_1 & +7x_2 & = & 3 \\ -5x_1 & +3x_2 & = & -8 \end{array}$$

Solution

The coefficient matrix is $A = \begin{bmatrix} 4 & 7 \\ -5 & 3 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 3 \\ -8 \end{bmatrix}$.

We get N_1 by replacing the first column of A with $\vec{b}$, and so $N_1 = \begin{bmatrix} 3 & 7 \\ -8 & 3 \end{bmatrix}$.

We get N_2 by replacing the second column of A with $\vec{b}$, and so $N_2 = \begin{bmatrix} 4 & 3 \\ -5 & -8 \end{bmatrix}$.

$\det A = \det\begin{bmatrix} 4 & 7 \\ -5 & 3 \end{bmatrix} = (4)(3) - (-5)(7) = 12 + 35 = 47$

$\det N_1 = \det\begin{bmatrix} 3 & 7 \\ -8 & 3 \end{bmatrix} = (3)(3) - (-8)(7) = 9 + 56 = 65$

$\det N_2 = \det\begin{bmatrix} 4 & 3 \\ -5 & -8 \end{bmatrix} = (4)(-8) - (-5)(3) = -32 + 15 = -17$

Cramer's Rule

Example

Use Cramer's Rule to solve this system of equations:

$$\begin{array}{rrcr} 4x_1 & +7x_2 & = & 3 \\ -5x_1 & +3x_2 & = & -8 \end{array}$$

Solution

$\det A = \det\begin{bmatrix} 4 & 7 \\ -5 & 3 \end{bmatrix} = (4)(3) - (-5)(7) = 12 + 35 = 47$

$\det N_1 = \det\begin{bmatrix} 3 & 7 \\ -8 & 3 \end{bmatrix} = (3)(3) - (-8)(7) = 9 + 56 = 65$

$\det N_2 = \det\begin{bmatrix} 4 & 3 \\ -5 & -8 \end{bmatrix} = (4)(-8) - (-5)(3) = -32 + 15 = -17$

So by Cramer's Rule, we have that

$$x_1 = \frac{\det N_1}{\det A} = \frac{65}{47} \qquad \text{and} \qquad x_2 = \frac{\det N_2}{\det A} = -\frac{17}{47}$$

We can verify that this is the solution to our system by plugging it in:

$$4\,\tfrac{65}{47} + 7\,\tfrac{-17}{47} = \tfrac{260-119}{47} = \tfrac{141}{47} = 3$$
$$-5\,\tfrac{65}{47} + 3\,\tfrac{-17}{47} = \tfrac{-325-51}{47} = \tfrac{-376}{47} = -8$$

Cramer's Rule

Example

Use Cramer's Rule to solve the system of equations

$$\begin{array}{rrrcr} 7x_1 & +2x_2 & -3x_3 & = & 6 \\ 4x_1 & +7x_2 & +6x_3 & = & 5 \\ 8x_1 & -9x_2 & -5x_3 & = & 4 \end{array}$$

Solution

The coefficient matrix is $A = \begin{bmatrix} 7 & 2 & -3 \\ 4 & 7 & 6 \\ 8 & -9 & -5 \end{bmatrix}$, and $\vec{b} = \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix}$.

We get N_1 by replacing the first column of A with $\vec{b}$, and so $N_1 = \begin{bmatrix} 6 & 2 & -3 \\ 5 & 7 & 6 \\ 4 & -9 & -5 \end{bmatrix}$.

We get N_2 by replacing the second column of A with $\vec{b}$, and so $N_2 = \begin{bmatrix} 7 & 6 & -3 \\ 4 & 5 & 6 \\ 8 & 4 & -5 \end{bmatrix}$.

We get N_3 by replacing the third column of A with $\vec{b}$, and so $N_3 = \begin{bmatrix} 7 & 2 & 6 \\ 4 & 7 & 5 \\ 8 & -9 & 4 \end{bmatrix}$.

Cramer's Rule

Example

Use Cramer's Rule to solve the system of equations

$$\begin{array}{cccll} 7x_1 & +2x_2 & -3x_3 & = & 6 \\ 4x_1 & +7x_2 & +6x_3 & = & 5 \\ 8x_1 & -9x_2 & -5x_3 & = & 4 \end{array}$$

Solution

$$\det A = \det\begin{bmatrix} 7 & 2 & -3 \\ 4 & 7 & 6 \\ 8 & -9 & -5 \end{bmatrix} = 7\begin{vmatrix} 7 & 6 \\ -9 & -5 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 8 & -5 \end{vmatrix} - 3\begin{vmatrix} 4 & 7 \\ 8 & -9 \end{vmatrix}$$

$$= 7(-35+54) - 2(-20-48) - 3(-36-56)$$

$$= 7(19) - 2(-68) - 3(-92) = 133 + 136 + 276$$

$$= 545$$

$$\det N_1 = \det\begin{bmatrix} 6 & 2 & -3 \\ 5 & 7 & 6 \\ 4 & -9 & -5 \end{bmatrix} = 6\begin{vmatrix} 7 & 6 \\ -9 & -5 \end{vmatrix} - 2\begin{vmatrix} 5 & 6 \\ 4 & -5 \end{vmatrix} - 3\begin{vmatrix} 5 & 7 \\ 4 & -9 \end{vmatrix}$$

$$= 6(-35+54) - 2(-25-24) - 3(-45-28)$$

$$= 6(19) - 2(-49) - 3(-73) = 114 + 98 + 219$$

$$= 431$$

Cramer's Rule

Example

Use Cramer's Rule to solve the system of equations

$$\begin{array}{cccll} 7x_1 & +2x_2 & -3x_3 & = & 6 \\ 4x_1 & +7x_2 & +6x_3 & = & 5 \\ 8x_1 & -9x_2 & -5x_3 & = & 4 \end{array}$$

Solution

$$\det N_2 = \det\begin{bmatrix} 7 & 6 & -3 \\ 4 & 5 & 6 \\ 8 & 4 & -5 \end{bmatrix} = 7\begin{vmatrix} 5 & 6 \\ 4 & -5 \end{vmatrix} - 6\begin{vmatrix} 4 & 6 \\ 8 & -5 \end{vmatrix} - 3\begin{vmatrix} 4 & 5 \\ 8 & 4 \end{vmatrix}$$

$$= 7(-25-24) - 6(-20-48) - 3(16-40)$$

$$= 7(-49) - 6(-68) - 3(-24) = -343 + 408 + 72$$

$$= 137$$

$$\det N_3 = \det\begin{bmatrix} 7 & 2 & 6 \\ 4 & 7 & 5 \\ 8 & -9 & 4 \end{bmatrix} = 7\begin{vmatrix} 7 & 5 \\ -9 & 4 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ 8 & 4 \end{vmatrix} + 6\begin{vmatrix} 4 & 7 \\ 8 & -9 \end{vmatrix}$$

$$= 7(28+45) - 2(16-40) + 6(-36-56)$$

$$= 7(73) - 2(-24) + 6(-92) = 511 + 48 - 552$$

$$= 7$$

Cramer's Rule

Example

Use Cramer's Rule to solve the system of equations

$$\begin{array}{rrrcl} 7x_1 & +2x_2 & -3x_3 & = & 6 \\ 4x_1 & +7x_2 & +6x_3 & = & 5 \\ 8x_1 & -9x_2 & -5x_3 & = & 4 \end{array}$$

Solution

And so we see that the solution to the system is

$$x_1 = \frac{\det N_1}{\det A} = \frac{431}{545}, \qquad x_2 = \frac{\det N_2}{\det A} = \frac{137}{545}, \qquad x_3 = \frac{\det N_3}{\det A} = \frac{7}{545}$$