Cross Product

Definition: The cross-product of vectors $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ is defined by

$$\vec{u} \times \vec{v} = \begin{bmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{bmatrix}$$

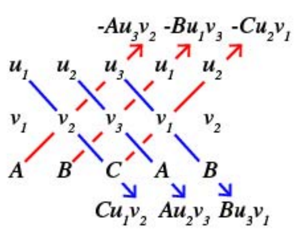
Note that the cross product is only defined in $\mathbb{R}^3$!

Example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \times \begin{bmatrix} 4 \\ -5 \\ 6 \end{bmatrix} = \begin{bmatrix} (2)(6) - (3)(-5) \\ (3)(4) - (1)(6) \\ (1)(-5) - (2)(4) \end{bmatrix} = \begin{bmatrix} 27 \\ 6 \\ -13 \end{bmatrix}$$

Cross Product

Calculating the Cross Product



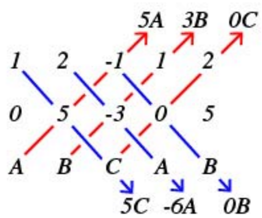
If we combine the A terms, we get $-Au_3v_2 + Au_2v_3 = A(u_2v_3 - u_3v_2)$.
Removing A, we have the first component of $\vec{u} \times \vec{v}$.
Similarly, $-Bu_1v_3 + Bu_3v_1$ leads us to the second compontent $u_3v_1 - u_1v_3$.
And $-Cu_2v_1 + Cu_1v_2$ leads us to the third component $u_1v_2 - u_2v_1$.

Cross Product

Example

Calculate the cross product of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix}$.

Solution



So our A terms are $5 - 6 = -1$, our B terms are $3 + 0 = 3$, and our C terms are $0 + 5 = 5$.

Thus, $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \times \begin{bmatrix} 0 \\ 5 \\ -3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix}$.

Cross Product

Theorem 1.5.1

For $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^3$ and $t \in \mathbb{R}$, we have

1. $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$
2. $\vec{x} \times \vec{x} = \vec{0}$
3. $\vec{x} \times (\vec{y} + \vec{z}) = \vec{x} \times \vec{y} + \vec{x} \times \vec{z}$
4. $(t\vec{x}) \times \vec{y} = t(\vec{x} \times \vec{y}) = \vec{x} \times (t\vec{y})$

Proof of $\vec{x} \times \vec{y} = -\vec{y} \times \vec{x}$

Suppose $\vec{x}, \vec{y} \in \mathbb{R}^3$. Then

$$\begin{aligned} -\vec{y} \times \vec{x} &= -\begin{bmatrix} y_2x_3 - y_3x_2 \\ y_3x_1 - y_1x_3 \\ y_1x_2 - y_2x_1 \end{bmatrix} \\ &= \begin{bmatrix} y_3x_2 - y_2x_3 \\ y_1x_3 - y_3x_1 \\ y_2x_1 - y_1x_2 \end{bmatrix} \\ &= \begin{bmatrix} x_2y_3 - x_3y_2 \\ x_3y_1 - x_1y_3 \\ x_1y_2 - x_2y_1 \end{bmatrix} \\ &= \vec{x} \times \vec{y} \end{aligned}$$

□

Cross Product

Proofs

Be sure to write out all the details of the proof, as we did above.

Every question is a proof!