Determining Linear Independence

Definition: A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is said to be linearly independent if the only solution to

$$\vec{0} = t_1\vec{v}_1 + \cdots + t_k\vec{v}_k$$

is $t_1 = t_2 = \cdots = t_k = 0$.

Determining Linear Independence

Example

Determine whether the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

Solution

To do this, we need to see if there are any parameters in the solution of the homogeneous system

$$\begin{array}{rcrcrcl} t_1 & + & t_2 & - & 3t_3 & = & 0 \\ 2t_1 & + & 4t_2 & - & 4t_3 & = & 0 \\ -t_1 & + & 7t_2 & & & = & 0 \end{array}$$

To determine this, we will row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & -3 \\ 2 & 4 & -4 \\ -1 & 7 & 0 \end{bmatrix} \begin{array}{l} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 2 & 2 \\ 0 & 8 & -3 \end{bmatrix} \begin{array}{l} \\ \frac{1}{2}R_2 \\ \\ \end{array} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 8 & -3 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 8R_1 \end{array} \sim \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & -11 \end{bmatrix}$$

This last matrix is in row echelon form, and thus we see that the rank of the coefficient matrix is 3. Since this is the same as the number of variables, there are no parameters in the general solution.

Thus, the set $\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} -3 \\ -4 \\ 0 \end{bmatrix} \right\}$ is linearly independent.

Determining Linear Independence

Example

Determine whether the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is linearly independent.

Solution

Again, this will come down to determining whether or not there are parameters in the solution of the homogeneous system

$$\begin{array}{rcrcrcl} & & t_2 & + & 3t_3 & = & 0 \\ 3t_1 & - & 7t_2 & & & = & 0 \\ -2t_1 & + & 6t_2 & + & 4t_3 & = & 0 \end{array}$$

To determine this, we will row reduce the coefficient matrix.

Determining Linear Independence

Example

Determine whether the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is linearly independent.

Solution

$$\begin{bmatrix} 0 & 1 & 3 \\ 3 & -7 & 0 \\ -2 & 6 & 4 \end{bmatrix} \begin{matrix} \\ \\ \frac{-1}{2} R_3 \end{matrix} \sim \begin{bmatrix} 0 & 1 & 3 \\ 3 & -7 & 0 \\ 1 & -3 & -2 \end{bmatrix} \begin{matrix} R_1 \updownarrow R_3 \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 3 & -7 & 0 \\ 0 & 1 & 3 \end{bmatrix} R_2 - 3R_1 \sim$$

$$\begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 1 & 3 \end{bmatrix} \begin{matrix} \\ \\ R_3 - \frac{1}{2} R_2 \end{matrix} \sim \begin{bmatrix} 1 & -3 & -2 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

This last matrix is in row echelon form, and thus we see that the rank of the coefficient matrix is 2. This means that there is one parameter in the general solution to the homogeneous system.

Thus, the set $\left\{ \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -7 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ in not linearly independent. That is, our set is linearly dependent.

Determining Linear Independence

Lemma 2.3.3

A set of vectors $\{\vec{v}_1, \ldots, \vec{v}_k\}$ in $\mathbb{R}^n$ is linearly independent if and only if the rank of the coefficient matrix of the homogeneous system $t_1\vec{v}_1 + \cdots + t_k\vec{v}_k = \vec{0}$ is k.

Theorem 2.3.4

If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$, then $k \leq n$.

In general, we actually use the theorem to determine that a set of vectors is linearly **dependent**.

Example

The set of vectors $\left\{ \begin{bmatrix} 2 \\ 7 \\ 5 \end{bmatrix}, \begin{bmatrix} 9 \\ 4 \\ -1 \end{bmatrix}, \begin{bmatrix} -8 \\ 13 \\ -4 \end{bmatrix}, \begin{bmatrix} -16 \\ 3 \\ 8 \end{bmatrix} \right\}$ is linearly dependent, since the set contains four vectors, and by Theorem 2.3.4, only sets with three or fewer vectors from $\mathbb{R}^3$ can be linearly independent.