Eigenvalues and Eigenvectors

Definition: Suppose that $L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping. A non-zero vector $\vec{v} \in \mathbb{R}^n$ such that $L(\vec{v}) = \lambda\vec{v}$ (for some real number λ) is called an eigenvector of L, and the scalar λ is called an eigenvalue of L. The pair $\lambda, \vec{v}$ is called an eigenpair.

Notes:

- L is a linear operator. This means that the standard matrix for L will be a square matrix.
- Eigenvectors must be non-zero. This is simply because we know $L(\vec{0}) = \vec{0}$ for any L, so there is nothing at all special about the fact that there is some $\lambda \in \mathbb{R}$ such that $L(\vec{0}) = \lambda\vec{0}$.
- While the vector $\vec{0}$ can not be an eigenvector, the real number 0 **can** be an eigenvalue, if we find a non-zero vector $\vec{v}$ such that $L(\vec{v}) = \vec{0} = 0\vec{v}$.

Eigenvalues and Eigenvectors

Example

Let $\vec{n} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and consider $\text{proj}_{\vec{n}} : \mathbb{R}^2 \to \mathbb{R}^2$. Then, given any vector $\vec{m} \in \mathbb{R}^2$, we see that

$$\text{proj}_{\vec{n}}(\vec{m}) = \frac{\vec{m} \cdot \vec{n}}{\|\vec{n}\|^2}\vec{n} = \frac{\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{1^2 + 0^2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{m_1}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} m_1 \\ 0 \end{bmatrix}$$

We can see that $\text{proj}_{\vec{n}}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is an eigenvector of $\text{proj}_{\vec{n}}$, with corresponding eigenvalue 1.

In general, since $\text{proj}_{\vec{n}}\left(\begin{bmatrix} m \\ 0 \end{bmatrix}\right) = \begin{bmatrix} m \\ 0 \end{bmatrix}$ for any $m \in \mathbb{R}$, the vectors $\begin{bmatrix} m \\ 0 \end{bmatrix}$ $(m \neq 0)$ are all eigenvectors of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 1.

Also, $\text{proj}_{\vec{n}}\left(\begin{bmatrix} 0 \\ m \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ for all $m \in \mathbb{R}$, so the vectors $\begin{bmatrix} 0 \\ m \end{bmatrix}$ $(m \neq 0)$ are also all eigenvectors of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 0.

Lastly, if the vectors $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$, with $m_1, m_2 \neq 0$, then $\text{proj}_{\vec{n}}\left(\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}\right) = \begin{bmatrix} m_1 \\ 0 \end{bmatrix}$. As $\begin{bmatrix} m_1 \\ 0 \end{bmatrix} \neq \lambda\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ for any $\lambda \in \mathbb{R}$, we see that $\begin{bmatrix} m_1 \\ m_2 \end{bmatrix}$ is not an eigenvector for $\text{proj}_{\vec{n}}$.

Eigenvalues and Eigenvectors

The textbook generalizes the previous example to $\text{proj}_{\vec{n}}$ for any vector $\vec{n} \in \mathbb{R}^n$.

Example

The result in the general case is that $\text{proj}_{\vec{n}}(s\vec{n}) = s\vec{n}$, so the vectors $s\vec{n}$ (for $s \neq 0$) are all eigenvectors of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 1.

If $\vec{v}$ is orthogonal to $\vec{n}$, then $\text{proj}_{\vec{n}}(s\vec{v}) = \vec{0}$, so the vectors $s\vec{v}$ (for $s \neq 0$) are all eigenvectors of $\text{proj}_{\vec{n}}$ with corresponding eigenvalue 0.

Only vectors that are either multiples of $\vec{n}$ or orthogonal to $\vec{n}$ are eigenvectors of $\text{proj}_{\vec{n}}$.

Eigenvalues and Eigenvectors

Example

Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping for a dilation by a factor of 5. That is, L is defined by $L(\vec{v}) = 5\vec{v}$.

Then every vector $\vec{v} \neq \vec{0}$ is an eigenvector of L, with corresponding eigenvalue 5.

Example

Consider $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$, where θ is **not** an integer multiple of π.

$\vec{w}$ is a scalar multiple of $\vec{v}$ if and only if they point in the same direction or opposite direction, but a rotation by our θ will not preserve direction.

Thus, we know that $R_\theta(\vec{v}) \neq s\vec{v}$ for any $s \in \mathbb{R}$, and so there are no eigenvectors of R_θ.

Definition: Suppose that A is an $n \times n$ matrix. A non-zero vector $\vec{v} \in \mathbb{R}^n$ such that $A\vec{v} = \lambda\vec{v}$ is called an eigenvector of A, and the scalar λ is called an eigenvalue of A. The pair $\lambda, \vec{v}$ is called an eigenpair.

Eigenvalues and Eigenvectors

Example

Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$.

Then we see that

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3+2 \\ 1+4 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector for A, with corresponding eigenvalue 5.

We also have that

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ -5 \end{bmatrix} = \begin{bmatrix} 30-10 \\ 10-20 \end{bmatrix} = \begin{bmatrix} 20 \\ -10 \end{bmatrix} = 2\begin{bmatrix} 10 \\ -5 \end{bmatrix}$$

so $\begin{bmatrix} 10 \\ -5 \end{bmatrix}$ is an eigenvector for A, with corresponding eigenvalue 2.

Finally, we look at:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6+2 \\ 2+4 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix} \neq \lambda\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

so $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is not an eigenvector for A.

Eigenvalues and Eigenvectors

Example

Let $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$. Then we look at

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+0+0+0 \\ -6+8-1-10 \\ 2+2-4+2 \\ 4+0+0+6 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \\ 2 \\ 10 \end{bmatrix} \neq \lambda\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$$

so $\begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}$ is not an eigenvector for B.

Next we can look at

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+0+0 \\ 0+4+1+0 \\ 0+1+4+0 \\ 0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 5 \\ 0 \end{bmatrix} = 5\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

so $\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ is an eigenvector for B, with corresponding eigenvalue 5.

Eigenvalues and Eigenvectors

Example

Let $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$. Then we look at

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ 5 \\ -5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0+0+0+0 \\ 0+20-5+0 \\ 0+5-20+0 \\ 0+0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 15 \\ -15 \\ 0 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 5 \\ -5 \\ 0 \end{bmatrix}$$

so $\begin{bmatrix} 0 \\ 5 \\ -5 \\ 0 \end{bmatrix}$ is an eigenvector for B, with corresponding eigenvalue 3.

Lastly, we'll look at

$$\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} 0 \\ -6 \\ 2 \\ 8 \end{bmatrix} = \begin{bmatrix} 0+0+0+0 \\ 0-24+2+40 \\ 0-6+8-8 \\ 0+0+0-24 \end{bmatrix} = \begin{bmatrix} 0 \\ 18 \\ -6 \\ -24 \end{bmatrix} = -3\begin{bmatrix} 0 \\ -6 \\ 2 \\ 8 \end{bmatrix}$$

so $\begin{bmatrix} 0 \\ -6 \\ 2 \\ 8 \end{bmatrix}$ is an eigenvector for B, with corresponding eigenvalue -3.