Elementary Matrices

Are elementary operations a linear mapping?
We would have to change our definition to allow inputs other than vectors, but the basic properties of linearity are that you preserve addition and scalar multiplication.

Example

Let L be the function whose domain and codomain are the set of 2×2 functions, and let L be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$$

Consider the following:

$$\begin{aligned} L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= L\left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}\right) \\ &= \begin{bmatrix} a_1+a_2+2c_1+2c_2 & b_1+b_2+2d_1+2d_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \\ &= \begin{bmatrix} a_1+2c_1 & b_1+2d_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2+2c_2 & b_2+2d_2 \\ c_2 & d_2 \end{bmatrix} \\ &= L\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + L\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \end{aligned}$$

Elementary Matrices

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We would have to change our definition to allow inputs other than vectors, but the basic properties of linearity are that you preserve addition and scalar multiplication.

Example

Let L be the function whose domain and codomain are the set of 2×2 functions, and let L be defined by

$$L\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$$

Consider the following:

$$\begin{aligned} L\left(s\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= L\left(\begin{bmatrix} sa & sb \\ sc & sd \end{bmatrix}\right) \\ &= \begin{bmatrix} sa+2sc & sb+2sd \\ sc & sd \end{bmatrix} \\ &= s\begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix} \\ &= sL\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \end{aligned}$$

So we see that the function L satisfies the linearity properties of preserving addition and scalar multiplication.

Elementary Matrices

Definition: A matrix that can be obtained from the identity matrix by a single elementary row operation is called an elementary matrix.

Note: An elementary matrix is the standard matrix for the row operation used to create it from the identity matrix.

Elementary Matrices

Example

$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is an elementary matrix, as it can be obtained from I_2 by adding 2 times the second row to the first row.

But notice that if we multiply $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ by a 2×2 matrix **on the left**,

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ c & d \end{bmatrix}$$

The resulting matrix is the same as if we had added 2 times the second row of our matrix to the first row.

This is the elementary row operation used to get $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ from I_2.

Elementary Matrices

Example

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is an elementary matrix, because it is obtained from I_3 by multiplying the third row by 5.

But notice that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ 5g & 5h & 5i \end{bmatrix}$$

So multiplying **on the left** by E maps a matrix to the row equivalent matrix you get by multiplying the third row by 5, just as how E was obtained from I_3 by multiplying the third row by 5.

Elementary Matrices

Example

$F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix, as it is the matrix you get when you switch the second and third rows of I_4.

We also see that

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a & b \\ e & f \\ c & d \\ g & h \end{bmatrix}$$

So multiplying by F **on the left** has the same effect as the elementary row operation of switching the second and third rows, just as F was obtained from I_4 by switching the second and third rows.

Elementary Matrices

You'll have noticed that I keep emphasizing that you multiply the elementary matrix **on the left**.
Since matrix multiplication is not commutative, you will not achieve the appropriate row operation if you multiply the elementary matrix on the right.

Example

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 2a+b \\ c & 2c+d \end{bmatrix}$ is not what you get if you add 2 times the second row of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to its first row.

Example

$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is not an elementary matrix, because you need two row operations to get A from I_3, namely $2R_1$, followed by $R_1 + R_3$.
Remember that replacing R_1 with $2R_1 + R_3$ **is not** an elementary row operation.
Replacing R_1 with $R_1 + 2R_3$ **is** an elementary row operation, which has matrix $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in $\mathbb{R}^3$.

Elementary Matrices

Theorem 3.6.1

If A is an $n \times n$ matrix and E is the elementary matrix obtained from I_n by a certain elementary row operation, then the product EA is the matrix obtained from A by performing the same elementary row operation.

Theorem 3.6.2

For any $m \times n$ matrix A, there exists a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \ldots E_2E_1A$ is equal to the reduced row echelon form of A.

Elementary Matrices

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 7 \\ -1 & -1 \end{bmatrix}$.

To find a sequence of elementary matrices $E_1, E_2, \ldots, E_k$ such that $E_k \ldots E_2 E_1 A$ is the reduced row echelon form of A, we first need to row reduce A to its reduced row echelon form.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 9 \\ -1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 3 \\ 0 & 3 \end{bmatrix} R_4 - R_3 \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} R_2 \updownarrow R_3 \sim \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tfrac{1}{3} R_2 \sim$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} R_1 - 2R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

It doesn't matter which order these row operations take place here, you always end up with the same matrix after you do all three of them.

There are lots of different ways to row reduce a matrix, so although the reduced row echelon form of a matrix is unique, the sequence $E_1, \ldots, E_k$ will not be unique.

As such, with these questions, it is especially important to write down your row operations, as that is the basis upon which your choices of $E_1, \ldots, E_k$ will be judged.

Elementary Matrices

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 7 \\ -1 & -1 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 9 \\ -1 & 1 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 + R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 3 \\ 0 & 3 \end{bmatrix} R_4 - R_3 \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} R_2 \updownarrow R_3 \sim \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \tfrac{1}{3} R_2 \sim$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} R_1 - 2R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Thus, $E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$,

$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, $E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, and $E_7 = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.