Finding Eigenvalues

We will need to find eigenvalues without even knowing what the eigenvectors are.

We know that λ is an eigenvalue for a matrix A if there is a non-zero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}$$

To solve this equation for λ, we can subtract $\lambda\vec{v}$ from both sides, getting

$$A\vec{v} - \lambda\vec{v} = \vec{0}$$

We can almost factor $\vec{v}$ out of this equation, but first we need to change the scalar λ into an $n \times n$ matrix.

Notice that $\lambda\vec{v}$ is the same as $(\lambda I)\vec{v}$, where I is the $n \times n$ identity matrix.

Using this, we see that

$$A\vec{v} = \lambda\vec{v} \text{ if and only if } (A - \lambda I)\vec{v} = \vec{0}$$

The equation $(A - \lambda I)\vec{v} = \vec{0}$ is a homogeneous system of linear equations with coefficient matrix $A - \lambda I$, and in this case we are looking for λ such that this system has a non-zero solution.

By the invertible matrix theorem, any system has a unique solution if and only if the determinant of the coefficient matrix is non-zero.

Since we do **not** want our system to have a unique solution, this means that we want the determinant of $A - \lambda I$ to be zero.

Summarizing, we have:

$$\lambda \text{ is an eigenvalue for } A \text{ if and only if } \det(A - \lambda I) = 0$$

Finding Eigenvalues

Example

Consider the matrix $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, which we saw previously.

We also saw that 2 and 5 were both eigenvalues for A.

We see now that these are the only eigenvalues for A by finding the eigenvalues for A as follows:

$$\begin{aligned} \det(A - \lambda I) &= \det\left(\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 3-\lambda & 2 \\ 1 & 4-\lambda \end{bmatrix}\right) \\ &= (3-\lambda)(4-\lambda) - (1)(2) \\ &= 12 - 3\lambda - 4\lambda + \lambda^2 - 2 \\ &= 10 - 7\lambda + \lambda^2 \\ &= (2-\lambda)(5-\lambda) \end{aligned}$$

We have that $\det(A - \lambda I) = 0$ if and only if $(2-\lambda)(5-\lambda) = 0$.

We see that λ is an eigenvalue for A if and only if $\lambda = 2, 5$.

Finding Eigenvalues

Example

Consider the matrix $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$, which we also saw previously.

We saw that 3, -3, and 5 were eigenvalues for B.

To find out if these are the only eigenvalues for B, we need to calculate the eigenvalues for B as follows:

$$\begin{aligned} \det(B - \lambda I) &= \det\left(\begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ -6 & 4-\lambda & 1 & 5 \\ 2 & 1 & 4-\lambda & -1 \\ 4 & 0 & 0 & -3-\lambda \end{bmatrix}\right) \\ &= (3-\lambda)\det\left(\begin{bmatrix} 4-\lambda & 1 & 5 \\ 1 & 4-\lambda & -1 \\ 0 & 0 & -3-\lambda \end{bmatrix}\right) \end{aligned}$$

$$\begin{aligned} &= (3-\lambda)(-3-\lambda)\det\left(\begin{bmatrix} 4-\lambda & 1 \\ 1 & 4-\lambda \end{bmatrix}\right) \\ &= (3-\lambda)(-3-\lambda)((4-\lambda)(4-\lambda)-(1)(1)) \\ &= (3-\lambda)(-3-\lambda)(16-8\lambda+\lambda^2-1) \\ &= (3-\lambda)(-3-\lambda)(15-8\lambda+\lambda^2) \\ &= (3-\lambda)(-3-\lambda)(3-\lambda)(5-\lambda) \\ &= (3-\lambda)^2(-3-\lambda)(5-\lambda) \end{aligned}$$

We see that $\det(B - \lambda I) = 0$ if and only if $(3-\lambda)^2(-3-\lambda)(5-\lambda) = 0$.

Thus we see that $\lambda = 3, -3, 5$ are the only eigenvalues for B.

These examples illustrate the fact that $\det(A - \lambda I)$ is a polynomial.

Finding Eigenvalues

Definition: Let A be an $n \times n$ matrix. Then $C(\lambda) = \det(A - \lambda I)$ is called the characteristic polynomial of A.

Note: This makes finding the eigenvalues of A the same as finding the solutions to $C(\lambda) = 0$.

Example

Previously, we found that the characteristic polynomial for $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ is $(2-\lambda)(5-\lambda)$ which expands to $10 - 7\lambda + \lambda^2$.

And the characteristic polynomial for $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$ is $(3-\lambda)^2(-3-\lambda)(5-\lambda)$ which expands to $-135 + 72\lambda + 6\lambda^2 - 8\lambda^3 + \lambda^4$.

As our only goal with the characteristic polynomial is to find its solutions, the factored form is preferable.