Finding Eigenvectors

Now that we know that eigenvalues are the solutions to $\det(A - \lambda I) = 0$, we can try to find the eigenvectors that correspond to these eigenvalues.
These vectors are the non-trivial solutions to the homogeneous system $(A - \lambda I)\vec{v} = \vec{0}$.
That is, the eigenvectors corresponding to the eigenvalue λ are the vectors in the solution space of the homogeneous system $(A - \lambda I)\vec{v} = \vec{0}$, not including the zero vector.
Instead of looking for specific eigenvectors, it is easier to solve for the solution space and then remove the zero vector.

Definition: Let λ be an eigenvalue of an $n \times n$ matrix A. Then the set containing the zero vector and all eigenvectors of A corresponding to λ is called the eigenspace of λ.

Finding Eigenvectors

Example

Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, as in the previous two lectures. We have seen that the eigenvalues of A are $\lambda = 2, 5$.
To find the eigenspace of $\lambda = 2$, we need to find the general solution to the homogeneous system $(A - 2I)\vec{v} = \vec{0}$:

$$A - 2I = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3-2 & 2 \\ 1 & 4-2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} R_2 - R_1 \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

So, our system is equivalent to the equation $v_1 + 2v_2 = 0$, or $v_1 = -2v_2$.
Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$$

Therefore, the eigenspace of $\lambda = 2$ is $\text{Span}\left\{\begin{bmatrix} -2 \\ 1 \end{bmatrix}\right\}$.

Note that this is consistent with our original example which showed that $\begin{bmatrix} 10 \\ -5 \end{bmatrix}$ is an eigenvector of A with eigenvalue 2.

Finding Eigenvectors

Example

Let $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$, as in the previous two lectures. We have seen that the eigenvalues of A are $\lambda = 2, 5$.

To find the eigenspace of $\lambda = 5$, we need to find the general solution to $(A - 5I)\vec{v} = \vec{0}$:

$$A - 5I = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3-5 & 2 \\ 1 & 4-5 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{matrix} -\frac{1}{2}R_1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \end{matrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So, our system is equivalent to the equation $v_1 - v_2 = 0$, or $v_1 = v_2$.
Replacing v_2 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

Thus, the eigenspace corresponding to $\lambda = 5$ is $\text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

Notes:

- This is consistent with our original example which showed that $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A, with eigenvalue 5.
- We looked at the matrix $A - \lambda I$ as we were finding the eigenvalues for A. So, instead of computing $A - 2I$ and $A - 5I$ from scratch, we could instead plug our values for λ into this matrix.

Finding Eigenvectors

Example

Let $B = \begin{bmatrix} 3 & 0 & 0 & 0 \\ -6 & 4 & 1 & 5 \\ 2 & 1 & 4 & -1 \\ 4 & 0 & 0 & -3 \end{bmatrix}$, as in the previous two lectures. The eigenvalues of B are $\lambda = 3, -3, 5$.

To find the eigenspaces of these eigenvalues, we need to look at the solutions to the homogeneous systems $(B - \lambda I)\vec{v} = \vec{0}$.
To do this, we will need to row reduce the coefficient matrices $B - \lambda I$.

When finding the eigenvalues of B, we saw that

$$B - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 0 & 0 \\ -6 & 4-\lambda & 1 & 5 \\ 2 & 1 & 4-\lambda & -1 \\ 4 & 0 & 0 & -3-\lambda \end{bmatrix}$$

So, to find the eigenspace of $\lambda = 3$, we need to row reduce the matrix

$$B - 3I = \begin{bmatrix} 3-3 & 0 & 0 & 0 \\ -6 & 4-3 & 1 & 5 \\ 2 & 1 & 4-3 & -1 \\ 4 & 0 & 0 & -3-3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -6 & 1 & 1 & 5 \\ 2 & 1 & 1 & -1 \\ 4 & 0 & 0 & -6 \end{bmatrix}$$

Finding Eigenvectors

Example

Row reducing, we get the following:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -6 & 1 & 1 & 5 \\ 2 & 1 & 1 & -1 \\ 4 & 0 & 0 & -6 \end{bmatrix} R_1 \updownarrow R_4 \sim \begin{bmatrix} 4 & 0 & 0 & -6 \\ -6 & 1 & 1 & 5 \\ 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tfrac{1}{4}R_1 \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ -6 & 1 & 1 & 5 \\ 2 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_2 + 6R_1 \\ R_3 - 2R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 1 & -4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 - R_2 \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tfrac{1}{6}R_3 \sim \begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1 + \tfrac{3}{2}R_3 \\ R_2 + 4R_3 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding Eigenvectors

Example

$$B - 3I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -6 & 1 & 1 & 5 \\ 2 & 1 & 1 & -1 \\ 4 & 0 & 0 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to

$$\begin{aligned} v_1 &= 0 \\ v_2 + v_3 &= 0 \\ v_4 &= 0 \end{aligned}$$

If we replace v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} 0 \\ -s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Thus, the eigenspace of $\lambda = 3$ is $\text{Span}\left\{\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}\right\}$

Finding Eigenvectors

Example

To find the eigenspace of $\lambda = -3$, we need to row reduce the matrix

$$B + 3I = \begin{bmatrix} 3+3 & 0 & 0 & 0 \\ -6 & 4+3 & 1 & 5 \\ 2 & 1 & 4+3 & -1 \\ 4 & 0 & 0 & -3+3 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 & 0 \\ -6 & 7 & 1 & 5 \\ 2 & 1 & 7 & -1 \\ 4 & 0 & 0 & 0 \end{bmatrix}$$

Row reducing, we get the following:

$$\begin{bmatrix} 6 & 0 & 0 & 0 \\ -6 & 7 & 1 & 5 \\ 2 & 1 & 7 & -1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \tfrac{1}{6}R_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -6 & 7 & 1 & 5 \\ 2 & 1 & 7 & -1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ R_2 + 6R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 5 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 \updownarrow R_3 \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 7 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 - 7R_2 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & -48 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix} -\tfrac{1}{48}R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 7 & -1 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 - 7R_3 \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding Eigenvectors

Example

$$B + 3I = \begin{bmatrix} 6 & 0 & 0 & 0 \\ -6 & 7 & 1 & 5 \\ 2 & 1 & 7 & -1 \\ 4 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3/4 \\ 0 & 0 & 1 & -1/4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to

$$\begin{aligned} v_1 \quad &= 0 \\ v_2 + (3/4)v_4 &= 0 \\ v_3 - (1/4)v_4 &= 0 \end{aligned}$$

If we replace v_4 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} 0 \\ (-3/4)s \\ (1/4)s \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ -3/4 \\ 1/4 \\ 1 \end{bmatrix} = \text{Span}\left\{\begin{bmatrix} 0 \\ -3/4 \\ 1/4 \\ 1 \end{bmatrix}\right\} = \text{Span}\left\{\begin{bmatrix} 0 \\ -3 \\ 1 \\ 4 \end{bmatrix}\right\}$$

Thus, the eigenspace of $\lambda = -3$ is $\text{Span}\left\{\begin{bmatrix} 0 \\ -3 \\ 1 \\ 4 \end{bmatrix}\right\}$.

Finding Eigenvectors

Example

To find the eigenspace of $\lambda = 5$, we need to row reduce the matrix

$$B - 5I = \begin{bmatrix} 3-5 & 0 & 0 & 0 \\ -6 & 4-5 & 1 & 5 \\ 2 & 1 & 4-5 & -1 \\ 4 & 0 & 0 & -3-5 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 0 & 0 \\ -6 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 4 & 0 & 0 & -8 \end{bmatrix}$$

Row reducing, we get the following:

$$\begin{bmatrix} -2 & 0 & 0 & 0 \\ -6 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 4 & 0 & 0 & -8 \end{bmatrix} -\tfrac{1}{2}R_1 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ -6 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 4 & 0 & 0 & -8 \end{bmatrix} \begin{matrix} \\ R_2 + 6R_1 \\ R_3 - 2R_1 \\ R_4 - 4R_1 \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & -8 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & -8 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4 + 2R_3 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ (-1)R_2 \\ \tfrac{1}{4}R_3 \\ \\ \end{matrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_2 + 5R_3 \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Finding Eigenvectors

Example

$$B - 5I = \begin{bmatrix} -2 & 0 & 0 & 0 \\ -6 & -1 & 1 & 5 \\ 2 & 1 & -1 & -1 \\ 4 & 0 & 0 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So our system is equivalent to

$$\begin{aligned} v_1 \quad &= 0 \\ v_2 - v_3 &= 0 \\ v_4 \quad &= 0 \end{aligned}$$

If we replace v_3 with the parameter s, we see that the general solution is

$$\vec{v} = \begin{bmatrix} 0 \\ s \\ s \\ 0 \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

Thus, the eigenspace of $\lambda = 5$ is $\operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}$.