Finding Inverses Using Elementary Matrices

Last lecture, we learned that for every matrix A, there is a sequence of elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1 A$ is the RREF of A.

Now, we consider the case where the RREF of A is I.
Then we have that $E_k \cdots E_1 A = I$.
But this means that $(E_k \cdots E_1)$ is A^{-1}.

Finding Inverses Using Elementary Matrices

Example

Let $A = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$. Consider the following row reduction of A to I:

$$\begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 5 & 8 \end{bmatrix} \begin{matrix} \\ R_2 - 5R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \begin{matrix} \\ -\frac{1}{2}R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 2R_2 \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we can construct a sequence of elementary matrices $E_4, \ldots, E_1$ such that $E_4 \cdots E_1 A = I$ as follows:

$$\frac{1}{2}R_1 \rightarrow \quad E_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$R_2 - 5R_1 \rightarrow \quad E_2 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}$$

$$-\frac{1}{2}R_2 \rightarrow \quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}$$

$$R_1 - 2R_2 \rightarrow \quad E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

Finding Inverses Using Elementary Matrices

Example

Then $A^{-1} = E_4E_3E_2E_1$, which we can calculate as follows:

$$E_2E_1 = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -5/2 & 1 \end{bmatrix}$$

$$E_3E_2E_1 = E_3(E_2E_1) = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ -5/2 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 5/4 & -1/2 \end{bmatrix}$$

$$A^{-1} = E_4E_3E_2E_1 = E_4(E_3E_2E_1) = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 0 \\ 5/4 & -1/2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5/4 & -1/2 \end{bmatrix}$$

We verify our calculation by looking at the product AA^{-1} :

$$AA^{-1} = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 5/4 & -1/2 \end{bmatrix} = \begin{bmatrix} -4+5 & 2-2 \\ -10+10 & 5-4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Finding Inverses Using Elementary Matrices

Example

Now let's look at a different row reduction of A to I:

$$\begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix} \; R_2 - 2R_1 \sim \begin{bmatrix} 2 & 4 \\ 1 & 0 \end{bmatrix} \; R_1 \updownarrow R_2 \sim \begin{bmatrix} 1 & 0 \\ 2 & 4 \end{bmatrix} \; R_2 - 2R_1 \sim \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \; \tfrac{1}{4}R_2 \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then we can construct a sequence of elementary matrices $E_4, \ldots, E_1$ such that $E_4 \cdots E_1A = I$ as follows:

$$R_2 - 2R_1 \rightarrow \quad E_1 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$R_1 \updownarrow R_2 \rightarrow \quad E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$R_2 - 2R_1 \rightarrow \quad E_3 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

$$\tfrac{1}{4}R_2 \rightarrow \quad E_4 = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}$$

Finding Inverses Using Elementary Matrices

Example

Then $A^{-1} = E_4E_3E_2E_1$, which we can calculate as follows:

$$E_2E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$$

$$E_3E_2E_1 = E_3(E_2E_1) = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$

$$A^{-1} = E_4E_3E_2E_1 = E_4(E_3E_2E_1) = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5/4 & -1/2 \end{bmatrix}$$

So we see that a different collection of row reduction steps will lead to a different sequence of matrices $E_k, \dots, E_1$. But no matter which sequence we use, we still end up with our unique matrix A^{-1} .

Finding Inverses Using Elementary Matrices

Now let's consider the inverse of an elementary matrix:

- Every elementary matrix must be row-equivalent to I and is therefore invertible.
- To get from an elementary matrix E to I, you need to undo the row operation you did to get from I to E.
- This will be a single row operation, and thus, the inverse of an elementary matrix is itself an elementary matrix.
- So the best way to find the inverse of an elementary matrix is to think in terms of row operations.

Finding Inverses Using Elementary Matrices

Example

$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$ since we undo $2R_1$ by performing $\frac{1}{2}R_1$.

In general, the inverse of the operation sR_i is $\dfrac{1}{s}R_i$.

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ since we undo switching R_1 and R_2 by switching them again.

In general, the inverse of the operation $R_i \updownarrow R_j$ is $R_i \updownarrow R_j$. That is, switching rows is its own inverse.

$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ since we undo $R_1 + 4R_3$ by performing $R_1 - 4R_3$.

In general, the inverse of the operation $R_i + sR_j$ is $R_i - sR_j$.

Finding Inverses Using Elementary Matrices

Example

Let $A = \begin{bmatrix} 2 & 4 \\ 5 & 8 \end{bmatrix}$, as in our earlier examples.

Then not only can we write A^{-1} as a product of elementary matrices, but we can also write A as a product of elementary matrices.

Since $A^{-1} = E_4E_3E_2E_1$, we have

$$A = (A^{-1})^{-1} = (E_4E_3E_2E_1)^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$$

Note: Recall that the order of multiplication switches when we distribute the inverse.

And since the inverse of an elementary matrix is itself an elementary matrix, we know that $E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}$ is a product of elementary matrices.

Specifically, we get that

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad \text{using the first row reduction}$$

or

$$A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \quad \text{using the second row reduction}$$

Finding Inverses Using Elementary Matrices

Theorem 3.6.3

If an $n \times n$ matrix A has rank n, then it may be represented as a product of elementary matrices.

Note:
When asked to "write A as a product of elementary matrices", you are expected to write out the matrices in full. Do not simply describe them using row operations, or leave them as E^{-1} even if you have already written out E.

Course Author's Theorem

If A is row equivalent to B, then there is an invertible matrix P such that $PA = B$.

Proof

If A is row equivalent to B, then there is a sequence of EROs taking A to B.
If $E_1, \dots, E_k$ are the elementary matrices for these row operations, then we have that $E_k \cdots E_1 A = B$.
If we let $P = E_k \cdots E_1$, then we have that P is invertible, since $E_1, \dots, E_k$ are invertible and the product of invertible matrices is always an invertible matrix.
Thus, P is an invertible matrix such that $PA = B$. $\square$