Finding the Inverse

Let A be an $n \times n$ matrix.
Then the search for A^{-1} is the search for an unknown matrix X such that $AX = I$.
If $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are the columns of X, then block multiplication turns the equation $AX = I$ into

$$\begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \cdots & \vec{e}_n \end{bmatrix}$$

Since two matrices are equal if and only if their columns are equal, this matrix equation is the same as the following list of systems:

$$A\vec{x}_1 = \vec{e}_1, \qquad A\vec{x}_2 = \vec{e}_2, \qquad \cdots, \qquad A\vec{x}_n = \vec{e}_n$$

In general, to find $\vec{x}_i$ such that $A\vec{x}_i = \vec{e}_i$, we would row reduce the augmented matrix $[\ A \mid \vec{e}_i\ ]$.
But, since all these systems have the same coefficient matrix (A), we would use the exact same row reduction steps to solve them all.
And since row operations take place within a column, we can actually solve all these systems simultaneously by looking at the following multi-augmented matrix:

$$\left[\ A \mid \vec{e}_1 \mid \vec{e}_2 \mid \cdots \mid \vec{e}_n\ \right]$$

If A is invertible, then Theorem 3.5.2 tells us that the rank of A is n, and thus the reduced row echelon form of A is I.
So, **if** A is invertible, when we row reduce we will end up with

$$\left[\ I \mid \vec{x}_1 \mid \vec{x}_2 \mid \cdots \mid \vec{x}_n\ \right]$$

Where the vectors $\vec{x}_i$ are our solutions to $A\vec{x}_i = \vec{e}_i$, and thus are the columns of $X = A^{-1}$.

Finding the Inverse

In general, when looking for the inverse of A, we drop the extra augmentation lines, and simply look at row reducing the matrix $[\ A \mid I\ ]$.
In the case when A is invertible, we will end up with $[\ I \mid A^{-1}\ ]$.
Also worth noting at this point is that if A has rank n, then $[\ A \mid I\ ]$ will row reduce to $[\ I \mid B\ ]$, and by the same arguments as above we know that $B = A^{-1}$, so now see that if A has rank n, then A is invertible.

But what if A isn't invertible?
Then at least one of our systems $A\vec{x}_i = \vec{e}_i$ does not have a solution.
That is, when we row reduce $[\ A \mid \vec{e}_i\ ]$, we would end up with a bad row.
And that means that the reduced row echelon form of A has at least one row of zeros.
So, to determine whether or not A is invertible, we need to see whether or not A has rank n, or equivalently to see whether or not the reduced row echelon form of A is I.

Algorithm

To find the inverse of a square matrix A:

1. Row reduce the multi-augmented matrix $[\ A \mid I\ ]$ so that the left block is in reduced row echelon form.
2. If the reduced row echelon form is $[\ I \mid B\ ]$, then $A^{-1} = B$.
3. If the reduced row echelon form of A is not I, then A is not invertible.

Finding the Inverse

Example

To determine if $A = \begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}$ is invertible, we will row reduce the matrix $\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right]$ as follows:

$$\left[\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] \begin{matrix} \frac{1}{3}R_1 \end{matrix} \sim \left[\begin{array}{cc|cc} 1 & 4/3 & 1/3 & 0 \\ 4 & 5 & 0 & 1 \end{array}\right] R_2 - 4R_1 \sim \left[\begin{array}{cc|cc} 1 & 4/3 & 1/3 & 0 \\ 0 & -1/3 & -4/3 & 1 \end{array}\right] -3R_2 \sim$$

$$\left[\begin{array}{cc|cc} 1 & 4/3 & 1/3 & 0 \\ 0 & 1 & 4 & -3 \end{array}\right] R_1 - \frac{4}{3}R_2 \sim \left[\begin{array}{cc|cc} 1 & 0 & -5 & 4 \\ 0 & 1 & 4 & -3 \end{array}\right]$$

From this, we see that $A^{-1} = \begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix}$.

We can (and should!) verify this by looking at $\begin{bmatrix} 3 & 4 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} -5 & 4 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} (3)(-5)+(4)(4) & (3)(4)+(4)(-3) \\ (4)(-5)+(5)(4) & (4)(4)+(5)(-3) \end{bmatrix}$
$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Finding the Inverse

Example

To determine if $B = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 8 \\ 2 & 1 & 1 & -5 \\ 2 & 2 & 8 & 6 \end{bmatrix}$ is invertible, we need to row reduce $[\ B \mid I\ ]$ as follows:

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 3 & 8 & 0 & 1 & 0 & 0 \\ 2 & 1 & 1 & -5 & 0 & 0 & 1 & 0 \\ 2 & 2 & 8 & 6 & 0 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ R_2 + R_1 \\ R_3 - 2R_1 \\ R_4 - 2R_1 \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & 4 & 8 & 1 & 1 & 0 & 0 \\ 0 & -3 & -1 & -5 & -2 & 0 & 1 & 0 \\ 0 & -2 & 6 & 6 & -2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \frac{1}{2}R_2 \\ \\ \\ \end{matrix} \sim$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 1/2 & 1/2 & 0 & 0 \\ 0 & -3 & -1 & -5 & -2 & 0 & 1 & 0 \\ 0 & -2 & 6 & 6 & -2 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ R_3 + 3R_2 \\ R_4 + 2R_2 \end{matrix} \sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 5 & 7 & -1/2 & 3/2 & 1 & 0 \\ 0 & 0 & 10 & 14 & -1 & 1 & 0 & 1 \end{array}\right] \begin{matrix} \\ \\ \\ R_4 - 2R_3 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 2 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 4 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 5 & 7 & -1/2 & 3/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & -2 & 1 \end{array}\right]$$

We do not need to row reduce further, because we have a row of all zeros in the left side.

This means that B is not row equivalent to I and thus B is not invertible.