## Gaussian Elimination

**Definition:** We say that two systems of equations are equivalent if they have the same solution set.

We want to make sure that whatever steps we take to solve a system, we end up with an equivalent system.

## Gaussian Elimination

### Solving Using Gaussian Elimination

**Step 1**: Multiply one equation by a non-zero constant.

**Example**

The following systems are equivalent:

$$\begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \end{aligned} \qquad \begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ x_1 + 2x_2 - 3x_3 - 5x_4 &= 7 \\ -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \end{aligned}$$

since we get the system on the right from the system on the left by multiplying the second equation by $1/2$.

It is easy to see why this step results in an equivalent system, as we get that $\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ is a solution to the equation $c(a_1x_1 + \cdots + a_nx_n) = cd$ if and only if it is also a solution to $a_1x_1 + \cdots + a_nx_n = d$, since we can simply multiply or divide by $c$.

We are simply replacing one equation with an equivalent equation, and thus not changing the overall solution set.

## Gaussian Elimination

### Solving Using Gaussian Elimination

**Step 2**: Interchange two equations. (That is, switch the positions of two equations.)

**Example**

The following systems are equivalent:

$$\begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \end{aligned} \qquad \begin{aligned} -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \end{aligned}$$

since we get the system on the right from the system on the left by swapping the first and third equations.

It should be obvious why this step results in an equivalent system.

## Gaussian Elimination

### Solving Using Gaussian Elimination

**Step 3**: Add a multiple of one equation to another equation.

**Example**

The following systems are equivalent:

$$\begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ -2x_1 - 13x_2 + 4x_3 + 4x_4 &= 25 \end{aligned} \qquad \begin{aligned} 3x_1 - 5x_2 + x_3 + 2x_4 &= -3 \\ 2x_1 + 4x_2 - 6x_3 - 10x_4 &= 14 \\ -9x_2 - 2x_3 - 6x_4 &= 39 \end{aligned}$$

since we get the system on the right from the system on the left by adding the second equation to the third.

We only need to show that the smaller system composed of the two involved equations is equivalent to the one we get by adding a multiple of one equation to the other.

## Gaussian Elimination

### Solving Using Gaussian Elimination

**Step 3**: Add a multiple of one equation to another equation.

Suppose we have the system

System 1

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= a \\ b_1x_1 + b_2x_2 + \cdots + b_nx_n &= b \end{aligned}$$

If we multiply the first equation by the constant $c$ and then add it to the second equation, we get

System 2

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= a \\ (ca_1 + b_1)x_1 + (ca_2 + b_2)x_2 + \cdots + (ca_n + b_n)x_n &= ca + b \end{aligned}$$

Our goal is to see that both of the above systems have the same solution set.

## Gaussian Elimination

### Solving Using Gaussian Elimination

**Step 3**: Add a multiple of one equation to another equation.

System 1

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= a \\ b_1x_1 + b_2x_2 + \cdots + b_nx_n &= b \end{aligned}$$

System 2

$$\begin{aligned} a_1x_1 + a_2x_2 + \cdots + a_nx_n &= a \\ (ca_1 + b_1)x_1 + (ca_2 + b_2)x_2 + \cdots + (ca_n + b_n)x_n &= ca + b \end{aligned}$$

We will first show that any solution of System 1 is also a solution of System 2 and then that any solution of System 2 is also a solution of System 1.

Assume $\vec{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ is a solution to System 1.

Plugging $\vec{s}$ into the equations gives

$$\begin{aligned} a_1s_1 + a_2s_2 + \cdots + a_ns_n &= a \\ b_1s_1 + b_2s_2 + \cdots + b_ns_n &= b \end{aligned}$$

Multiplying the first equation by $c$ gives

$$ca_1s_1 + ca_2s_2 + \cdots + ca_ns_n = ca$$

Adding this result to the second equation gives

$$ca_1s_1 + ca_2s_2 + \cdots + ca_ns_n + b_1s_1 + b_2s_2 + \cdots + b_ns_n = ca + b$$

Rearranging this gives

$$(ca_1 + b_1)s_1 + (ca_1 + b_2)s_2 + \cdots + (ca_n + b_n)s_n = ca + b$$

Thus, $\vec{s}$ is a solution to System 2.

# Gaussian Elimination

## Solving Using Gaussian Elimination

**Step 3**: Add a multiple of one equation to another equation.

| System 1 | System 2 |
|---|---|
| $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a$ | $a_1x_1 + a_2x_2 + \cdots + a_nx_n = a$ |
| $b_1x_1 + b_2x_2 + \cdots + b_nx_n = b$ | $(ca_1 + b_1)x_1 + (ca_2 + b_2)x_2 + \cdots + (ca_n + b_n)x_n = ca + b$ |

We will first show that any solution of System 1 is also a solution of System 2 and then that any solution of System 2 is also a solution of System 1.

Assume $\vec{t} = \begin{bmatrix} t_1 \\ \vdots \\ t_n \end{bmatrix}$ is a solution to System 2.

Plugging $\vec{t}$ into the equations of System 2 gives

$$\begin{aligned} a_1t_1 + a_2t_2 + \cdots + a_nt_n &= a \\ (ca_1 + b_1)t_1 + (ca_1 + b_2)t_2 + \cdots + (ca_n + b_n)t_n &= ca + b \end{aligned}$$

Multiplying the first equation by $c$ gives

$$ca_1t_1 + ca_2t_2 + \cdots + ca_nt_n = ca$$

Subtracting this result from the second question gives

$$b_1t_1 + b_2t_2 + \cdots + b_nt_n = b$$

Thus $\vec{t}$ is a solution of System 1.

**Note:** The third step is known as the elimination method as we want to eliminate as many variables as possible from each equation.

# Gaussian Elimination

## Example

Find the solution set for the following system of equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 11 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

**Solution**

Our first goal will be to eliminate the variable $x_1$ from two of the three equations.

We can eliminate $x_1$ from the second equation by adding the first equation to the second equation, giving us the equivalent system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & 5x_2 & + & 3x_3 & = & 7 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

## Gaussian Elimination

### Example

Find the solution set for the following system of equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 11 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

which is equivalent to

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & 5x_2 & + & 3x_3 & = & 7 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

**Solution**

We can remove the variable $x_1$ from the third equation by adding $-2$ times the first equation to the third equation. This calculation can be written as

$$\begin{array}{rrcrcrcr} & -2(x_1 & + & 2x_2 & - & x_3 & = & -4) \\ \hline & -2x_1 & - & 4x_2 & + & 2x_3 & = & 8 \\ + & 2x_1 & + & 5x_2 & + & x_3 & = & 3 \\ \hline & & & x_2 & + & 3x_3 & = & 11 \end{array}$$

So our new system is

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & 5x_2 & + & 3x_3 & = & 7 \\ & & x_2 & + & 3x_3 & = & 11 \end{array}$$

## Gaussian Elimination

### Example

Find the solution set for the following system of equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ -x_1 & + & 3x_2 & + & 4x_3 & = & 11 \\ 2x_1 & + & 5x_2 & + & x_3 & = & 3 \end{array}$$

which is equivalent to

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & 5x_2 & + & 3x_3 & = & 7 \\ & & x_2 & + & 3x_3 & = & 11 \end{array}$$

**Solution**

Our second goal is to remove the $x_2$ variable from one of the equations that already has the $x_1$ variable removed, leaving only the $x_3$ variable.

We add $-5$ times the third equation to the second equation.

The calculation is

$$\begin{array}{rrcrcr} & -5(x_2 & + & 3x_3 & = & 11) \\ \hline & -5x_2 & - & 15x_3 & = & -55 \\ + & 5x_2 & + & 3x_3 & = & 7 \\ \hline & & - & 12x_3 & = & -48 \end{array}$$

So our new system is

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -4 \\ & & & - & 12x_3 & = & -48 \\ & & x_2 & + & 3x_3 & = & 11 \end{array}$$

By back-substitution, as shown in a previous lecture, we get that the general solution is the single vector $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$.

## Gaussian Elimination

### Example

Find the solution set for the system

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13 \\ 2x_1 & + & 4x_2 & - & 4x_3 & + & 8x_4 & = & 26 \\ 2x_1 & + & 4x_2 & - & 3x_3 & + & 5x_4 & = & 18 \end{array}$$

**Solution**

To eliminate $x_1$ from two of the equations we add $-2$ times the first equation to each of the second and third equations. The calculations are

$$\begin{array}{rrcrcrcrcr} & -2(x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13) \\ \hline & -2x_1 & - & 4x_2 & + & 4x_3 & - & 8x_4 & = & -26 \\ + & 2x_1 & + & 4x_2 & - & 4x_3 & + & 8x_4 & = & 26 \\ \hline & & & & & & & 0 & = & 0 \end{array} \qquad \begin{array}{rrcrcrcrcr} & -2(x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13) \\ \hline & -2x_1 & - & 4x_2 & + & 4x_3 & - & 8x_4 & = & -26 \\ + & 2x_1 & + & 4x_2 & - & 3x_3 & + & 5x_4 & = & 18 \\ \hline & & & & & x_3 & - & 3x_4 & = & -8 \end{array}$$

Thus, our new system is

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & - & 2x_3 & + & 4x_4 & = & 13 \\ & & & & & & 0 & = & 0 \\ & & & & x_3 & - & 3x_4 & = & -8 \end{array}$$

By back-substitution, as demonstrated in the previous lecture, we get that the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ -8 \\ 0 \end{bmatrix} + s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix},\ s, t \in \mathbb{R}$$

## Gaussian Elimination

### Example

Find the general solution to the system

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & + & x_3 & = & -8 \\ x_1 & + & x_2 & - & x_3 & = & -2 \\ -x_1 & + & 2x_2 & + & 7x_3 & = & 7 \end{array}$$

**Solution**

The first thing we want to do is eliminate $x_1$ from two of our equations.

We can interchange the first and second equation to get the equivalent system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ 3x_1 & + & 5x_2 & + & x_3 & = & -8 \\ -x_1 & + & 2x_2 & + & 7x_3 & = & 7 \end{array}$$

Now we can add $-3$ times the first equation to the second equation, and add the first equation to the third equation, giving us the equivalent system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ & & 2x_2 & + & 4x_3 & = & -2 \\ & & 3x_2 & + & 6x_3 & = & 5 \end{array}$$

## Gaussian Elimination

### Example

Find the general solution to the system

$$\begin{array}{rcrcrcr} 3x_1 & + & 5x_2 & + & x_3 & = & -8 \\ x_1 & + & x_2 & - & x_3 & = & -2 \\ -x_1 & + & 2x_2 & + & 7x_3 & = & 7 \end{array}$$

which is equivalent to

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ & & 2x_2 & + & 4x_3 & = & -2 \\ & & 3x_2 & + & 6x_3 & = & 5 \end{array}$$

**Solution**

We see now that $x_2$ will be a leading variable, so now our goal is to eliminate $x_2$ from either the second or third equation.

Notice that all the coefficients in the second equation are a multiple of $2$, so we can easily multiply equation two by $1/2$ to get the system

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ & & x_2 & + & 2x_3 & = & -1 \\ & & 3x_2 & + & 6x_3 & = & 5 \end{array}$$

Now we can add $-3$ times equation two to equation three, getting

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & -2 \\ & & x_2 & + & 2x_3 & = & -1 \\ & & & & 0 & = & 8 \end{array}$$

Since $0 = 8$ is a contradiction, we see that our system has no solutions.