Geometrical Transformations

Previously, we defined linear mappings and noted that the mappings $\text{proj}_{\vec{n}}$ and $\text{perp}_{\vec{n}}$ were examples of linear mappings.

This lecture will introduce you to some other commonly other used linear mappings which have visual interpretation as some kind of geometrical transformation.

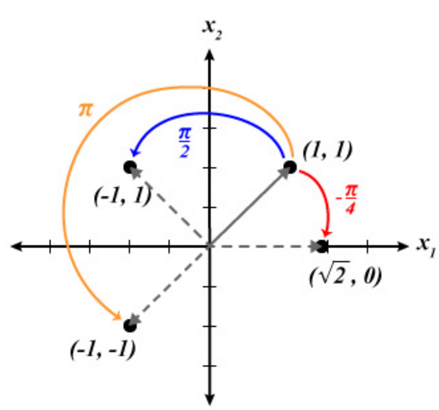
The results of the text will be summarized so you can focus on the calculations needed for geometrical transformations.

You will not need to prove any of these transformations are linear mappings, to prove that the indicated matrices actually correspond to the affiliated transformation, or to graph any of the transformations.

Rotations Through θ in $\mathbb{R}^2$

Definition: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is defined to be the transformation that rotates $\vec{x}$ counterclockwise through angle θ to the image $R_\theta(\vec{x})$. The standard matrix for R_θ is $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Example



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Example

The results in the diagrams can be verified using the standard matrix for a rotation.

$$R_{\pi/2}(1,1) = \begin{bmatrix} \cos\left(\frac{\pi}{2}\right) & -\sin\left(\frac{\pi}{2}\right) \\ \sin\left(\frac{\pi}{2}\right) & \cos\left(\frac{\pi}{2}\right) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$R_{\pi}(1,1) = \begin{bmatrix} \cos(\pi) & -\sin(\pi) \\ \sin(\pi) & \cos(\pi) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$R_{-\pi/4}(1,1) = \begin{bmatrix} \cos\left(-\frac{\pi}{4}\right) & -\sin\left(-\frac{\pi}{4}\right) \\ \sin\left(-\frac{\pi}{4}\right) & \cos\left(-\frac{\pi}{4}\right) \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix}$$

Stretches and Shears

Definition: For $t \in \mathbb{R}$, $t > 0$, a stretch by a factor of t in the x_i direction means to multiply the x_i term by t, but leave all other terms unchanged. Visually, we are pulling $\vec{x}$ in the x_i direction, but the amount of pulling depends on the distance of $\vec{x}$ from the origin (approximated by the x_i term). If $t < 1$, this is is sometimes referred to as a shrink instead of a stretch. The matrix for a stretch is obtained by replacing the 1 in the ii-th term of the identity matrix with t.

Example

In $\mathbb{R}^2$, the matrix for a stretch by a factor of 2 in the x_2 direction is $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

In $\mathbb{R}^4$, the matrix for a stretch by a factor of 2 in the x_2 direction is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Stretches and Shears

A shear is similar to a stretch, but instead of pulling the x_i term in the x_i direction, we pull the x_i term in the x_j direction, where $i \neq j$.

Definition: For $s \in \mathbb{R}$, a shear of in the x_i by a factor of sx_j means to "push" $\vec{x}$ in the x_i direction by sx_j (where $j \neq i$). Thus, the amount of shear applied to $\vec{x}$ depends both on s and on how far $\vec{x}$ is from the origin (which is approximated by x_j). The matrix for a shear is obtained by replacing the 0 in the ij-th term of the identity matrix with s.

Example

In $\mathbb{R}^2$, the matrix for a shear by a factor of $-3x_1$ in the x_2 direction is $\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

Since $i = 2$ and $j = 1$, we replace the 21-term in the identity matrix with -3.

In $\mathbb{R}^3$, the matrix for a shear by a factor of $8x_3$ in the x_2 direction is $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 8 \\ 0 & 0 & 1 \end{bmatrix}$.

This is the same as the function $S(x_1, x_2, x_3) = (x_1, x_2 + 8x_3, x_3)$.

Stretches and Shears

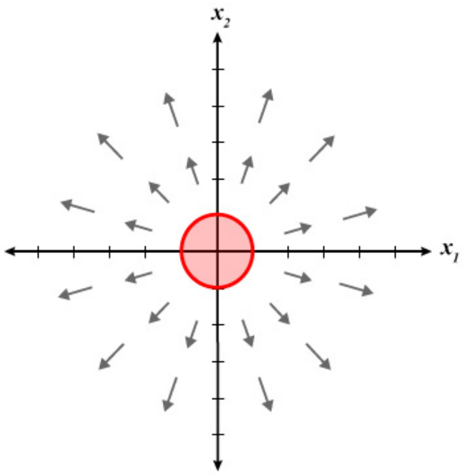
An idea related to stretches and shears is a translation. A translation is when you move an object to a different location. Mathematically, this is achieved by adding a fixed vector to all your vectors.

Example

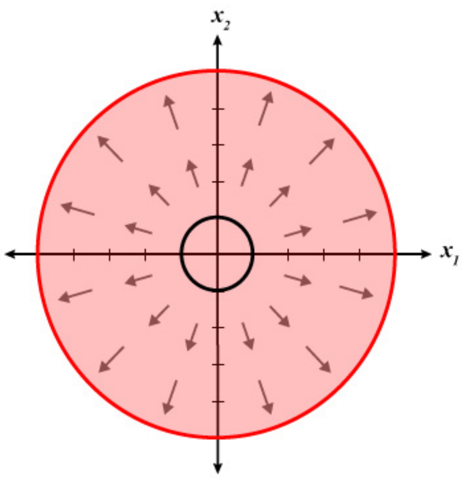
$T(x_1, x_2) = (x_1, x_2) + (2, 3)$ would define a translation. It can also be written as $T(x_1, x_2) = (x_1 + 2, x_2 + 3)$.

Note: Translations aren't mentioned in the text as they are not linear mappings. They are mentioned here to emphasize the difference between a translation like $T(x_1, x_2) = (x_1 + 2, x_2 + 3)$, which is not linear, and a stretch like $S_1(x_1, x_2) = (2x_1, x_2)$ or a shear like $S_2(x_1, x_2) = (x_1 + 3x_2, x_2)$ both of which are linear mappings.

Contractions and Dilations



Contractions and Dilations



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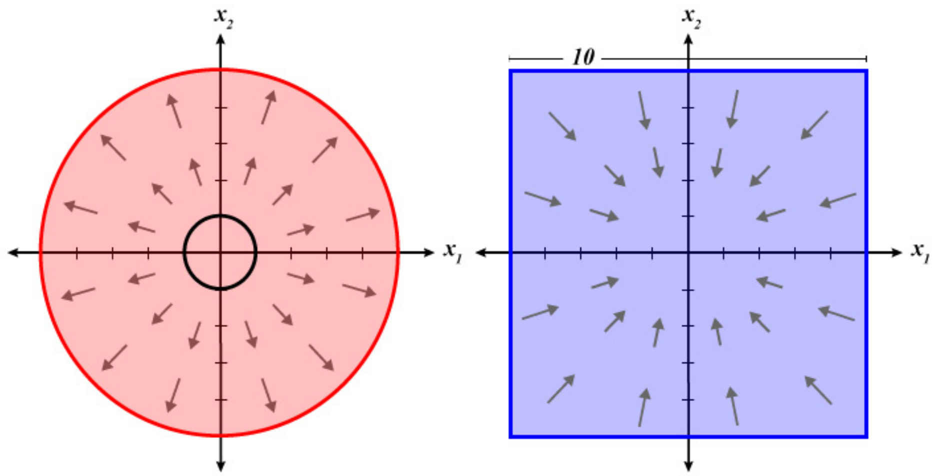

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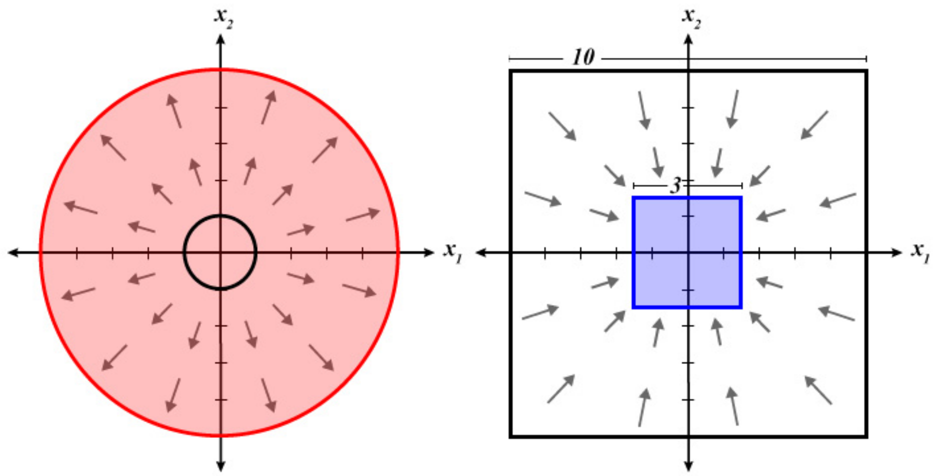


Contractions and Dilations

Definition: For $t \in \mathbb{R}$, $t > 1$, the dilation of $\vec{x}$ by a factor of t is the function $T(\vec{x}) = t\vec{x}$. If $0 < t < 1$, the function $T(\vec{x}) = t\vec{x}$ is called the contraction of $\vec{x}$ by a factor of t. As these are the same function, they have the same standard matrix, which is obtained by multiplying the identity matrix by t.

Examples

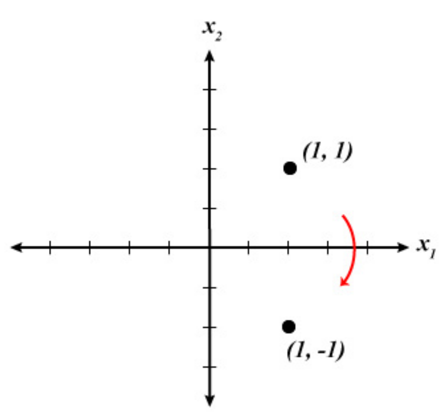
A dilation by a factor of 3 in $\mathbb{R}^3$ has matrix $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

A contraction by a factor of $\frac{2}{7}$ in $\mathbb{R}^2$ has matrix $\begin{bmatrix} \frac{2}{7} & 0 \\ 0 & \frac{2}{7} \end{bmatrix}$.

Reflections

The easiest reflection to consider is a reflection across a coordinate axis in $\mathbb{R}^2$.

Example



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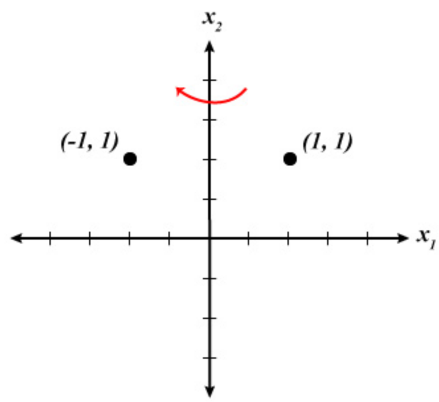

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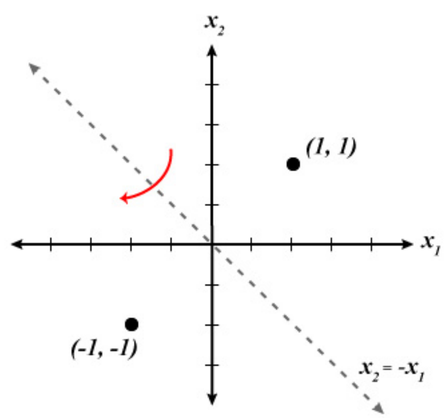
Example



In general, we can reflect across any plane through the origin in $\mathbb{R}^3$, and through any hyperplane through the origin in any $\mathbb{R}^n$.

All of these objects can be described by an equation of the form $\vec{n} \cdot \vec{x} = 0$. The normal vector $\vec{n}$ to describes the location of an arbitrary point $\vec{p}$ relative to the line $\vec{n} \cdot \vec{x} = 0$, and that information can be used to plot its new location after the reflection.

Reflections

Definition: Let $\vec{n} \cdot \vec{x} = 0$ define a line (or a plane) through the origin in $\mathbb{R}^2$ (or $\mathbb{R}^3$). A reflection in the line/plane with normal vector $\vec{n}$ will be denoted $\text{refl}_{\vec{n}}$, and we have that

$$\text{refl}_{\vec{n}}(\vec{p}) = \vec{p} - 2\text{proj}_{\vec{n}}(\vec{p})$$

Reflections

Example

Determine the matrix of the reflection in the line $2x_1 + 3x_2 = 0$ in $\mathbb{R}^2$.

Solution

We need to compute $\text{refl}_{\vec{n}}\vec{e}_1$ and $\text{refl}_{\vec{n}}\vec{e}_2$. To do this, the very first thing we need to do is read off the fact that $\vec{n} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ from our equation. Next, in order to compute the necessary $\text{refl}_{\vec{n}}$ values, we first need to compute the related $\text{proj}_{\vec{n}}$ values. To that end, we first note the following calculations:

$$||\vec{n}||^2 = 2^2 + 3^2 = 13 \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2 \qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 3$$

Then we see that

$$\text{proj}_{\vec{n}}\vec{e}_1 = \frac{\vec{e}_1 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{2}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 4/13 \\ 6/13 \end{bmatrix}$$
$$\text{proj}_{\vec{n}}\vec{e}_2 = \frac{\vec{e}_2 \cdot \vec{n}}{||\vec{n}||^2}\vec{n} = \frac{3}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 6/13 \\ 9/13 \end{bmatrix}$$

Reflections

Example

Determine the matrix of the reflection in the line $2x_1 + 3x_2 = 0$ in $\mathbb{R}^2$.

Solution

Now we can compute

$$\text{refl}_{\vec{n}}\vec{e}_1 = \vec{e}_1 - 2\text{proj}_{\vec{n}}\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 4/13 \\ 6/13 \end{bmatrix} = \begin{bmatrix} 5/13 \\ -12/13 \end{bmatrix}$$
$$\text{refl}_{\vec{n}}\vec{e}_2 = \vec{e}_2 - 2\text{proj}_{\vec{n}}\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 6/13 \\ 9/13 \end{bmatrix} = \begin{bmatrix} -12/13 \\ -5/13 \end{bmatrix}$$

Thus, we have that $[\text{refl}_{\vec{n}}] = \begin{bmatrix} 5/13 & -12/13 \\ -12/13 & -5/13 \end{bmatrix}$.