Homogeneous Linear Equations

Definition: A linear equation is homogeneous if the right-hand side is zero. A system of linear equations is homogeneous if all of the equations of the system are homogeneous.

Homogeneous Linear Equations

Example

Find the general solution of the homogeneous system

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 6x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & + & 6x_2 & + & 9x_3 & = & 0 \end{array}$$

Solution

Before we even put this into matrix form, there is one solution that is immediately obvious: $\vec{x} = \vec{0}$. In fact, $\vec{0}$ will be a solution to every homogeneous system, and it is so obvious a solution that it is called the trivial solution.

We do not need to ask whether or not the system is consistent, as every homogeneous system is consistent. But, one solution is not necessarily the whole solution, so let's go ahead and turn our system into an augmented matrix and row reduce.

Homogeneous Linear Equations

Example

Find the general solution of the homogeneous system

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 6x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & + & 6x_2 & + & 9x_3 & = & 0 \end{array}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \begin{array}{c} R_1 \updownarrow R_2 \\ \\ \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 4 & 6 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \begin{array}{c} \\ \frac{1}{4} R_2 \\ \\ \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \\ R_3 - 6R_2 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Homogeneous Linear Equations

Example

Find the general solution of the homogeneous system

$$\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 6x_3 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & = & 0 \\ 3x_1 & + & 6x_2 & + & 9x_3 & = & 0 \end{array}$$

Solution

$$\left[\begin{array}{ccc|c} 2 & 4 & 6 & 0 \\ 1 & 2 & 1 & 0 \\ 3 & 6 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Turning our RREF matrix back into equations, we have

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & & & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

We need to replace the variable x_2 with the parameter s, giving us

$$\begin{array}{rcrcrcl} x_1 & + & 2s & & & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

From this we see that the general solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s \\ s \\ 0 \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$

Note: The last column will always be a column of zeros. For that reason, people usually drop the augmented column, and focus only on the coefficient matrix, when they are dealing with a homogeneous system.

Homogeneous Linear Equations

Example

Find the general solution of the homogeneous system

$$\begin{array}{rcrcl} x_1 & + & 7x_2 & = & 0 \\ -3x_1 & - & 3x_2 & = & 0 \end{array}$$

Solution

We will row reduce the coefficient matrix, as follows:

$$\begin{bmatrix} 1 & 7 \\ -3 & -3 \end{bmatrix} \begin{matrix} \\ R_2 + 3R_1 \end{matrix} \sim \begin{bmatrix} 1 & 7 \\ 0 & 18 \end{bmatrix} \begin{matrix} \\ \frac{1}{18} R_2 \end{matrix} \sim \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \begin{matrix} R_1 - 7R_2 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Turning our RREF matrix back into equations, we have

$$\begin{array}{rcl} x_1 & = & 0 \\ x_2 & = & 0 \end{array}$$

From this we see that the only solution is $\vec{x} = \vec{0}$.