Invertible Matrices

There are some matrices that we can "divide" by.
There are many ways to look at division, but for matrices we want to look at the idea of a multiplicative inverse.

Definition: Let A be an $n \times n$ matrix. If there exists an $n \times n$ matrix B such that $AB = I = BA$, then A is said to be invertible, and B is called the inverse of A (and A is the inverse of B). The inverse of A is denoted A^{-1}.

Example

$\frac{1}{3}$ is the inverse of 3, since $(3)\left(\frac{1}{3}\right) = 1 = \left(\frac{1}{3}\right)(3)$

When you say "$a \div b$", you are saying the same thing as "$a \times b^{-1}$".

Notes:

- Our definition of a matrix inverse only applies to square matrices, so this already rules out a general definition of matrix division.
- Some $n \times n$ matrices do not have an inverse.

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Example

$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$ is the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, because

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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Example

The matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ does not have an inverse.

To see this, suppose by way of contradiction that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ does have an inverse $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\begin{aligned} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} a+c & b+d \\ a+c & b+d \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

This means that the components will be $a+c=1$, $a+c=0$, $b+d=0$, $b+d=1$.

Since $a+c=1$ and $a+c=0$ is a contradiction (as is $b+d=0$ and $b+d=1$), we see that $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ is not an invertible matrix.

Invertible Matrices

We have been referring to "the" inverse of A. The fact that a matrix has only one inverse is proved as follows:

Theorem 3.5.1

Let A be a square matrix and suppose that $BA = AB = I$ and $CA = AC = I$. Then $B = C$.

Proof

We have $B = BI = B(AC) = (BA)C = IC = C$. □

Note: The definition of the inverse says that we need $AB = I$ *and* $BA = I$. Since matrix multiplication is not commutative, it is important to list both of these conditions.

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Theorem 3.5.2

Suppose that A and B are $n \times n$ matrices such that $AB = I$. Then $BA = I$, so that $B = A^{-1}$. Moreover, B and A have rank n.

Proof

To show that $BA = I$, we will use the fact that if $(BA)\vec{y} = I\vec{y}$ for all $\vec{y} \in \mathbb{R}^n$, then $BA = I$. But, as $I\vec{y} = \vec{y}$, we will in fact aim to show that $(BA)\vec{y} = \vec{y}$ for all $\vec{y} \in \mathbb{R}^n$.
The first step in this process will be to start at the ending, that is, to show that B has rank n.

We will prove this by contradiction, so let us assume that B does **not** have rank n. Then the homogeneous system $B\vec{x} = \vec{0}$ has a non-trivial solution.
This means that there is some non-zero vector $\vec{y}$ such that $B\vec{y} = \vec{0}$, but this also means that

$$\vec{y} = I\vec{y} = (AB)\vec{y} = A(B\vec{y}) = A\vec{0} = \vec{0}$$

which is a contradiction.
We have therefore shown that B has rank n.

Invertible Matrices

Theorem 3.5.2

Suppose that A and B are $n \times n$ matrices such that $AB = I$. Then $BA = I$, so that $B = A^{-1}$. Moreover, B and A have rank n.

Proof

Since B has rank n, we know that the system of equations $B\vec{x} = \vec{y}$ is consistent for all $\vec{y} \in \mathbb{R}^n$. This means that for any $\vec{y} \in \mathbb{R}^n$, there is some $\vec{z} \in \mathbb{R}^n$ such that $B\vec{z} = \vec{y}$.
For every $\vec{y} \in \mathbb{R}^n$, we have

$$(BA)\vec{y} = (BA)(B\vec{z}) = B((AB)\vec{z}) = B(I\vec{z}) = B\vec{z} = \vec{y}$$

We have now shown that $(BA)\vec{y} = \vec{y}$ for all $\vec{y} \in \mathbb{R}^n$, and thus that $BA = I$.

Now that we have $BA = I$, we can also prove that A has rank n using the same argument that we used to show that B has rank n. □

Note: We also get that if $BA = I$, then $AB = I$ and thus that $B = A^{-1}$, by simply reversing the roles of A and B in Theorem 3.5.2, and using the fact that if $A = B^{-1}$, then $B = A^{-1}$.

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Theorem 3.5.3

Suppose that A and B are invertible matrices and that t is a non-zero real number.

(a) $(tA)^{-1} = \frac{1}{t}A^{-1}$
(b) $(AB)^{-1} = B^{-1}A^{-1}$
(c) $(A^T)^{-1} = (A^{-1})^T$

Proof

To show that $C = A^{-1}$, we need only show that $AC = I$.
These proofs also make use of a variety of previous "useful property" theorems.

(a): $(tA)(\frac{1}{t}A^{-1}) = (\frac{1}{t})(tAA^{-1}) = \frac{t}{t}I = I$

(b): $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$

(c): $(A)^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$ (This uses the fact that $(AB)^T = B^TA^T$.)

□

Invertible Matrices

Suppose that A is an $n \times m$ matrix.
There could then be an $m \times n$ matrix such that $AB = I$ and $BA = I$, but you wouldn't have $AB = BA$, since AB is an $n \times n$ matrix and BA is an $m \times m$ matrix.

We thus break the idea of an inverse into a left inverse (an $m \times n$ matrix B such that $BA = I$) and a right inverse (an $m \times n$ matrix C such that $AC = I$).

Theorem 3.5.2 explains why it isn't necessary to bother with such a distinction for square matrices, but if $m \neq n$ all sorts of crazy things can happen:

- You can have a left inverse but no right inverse.
- You can have a right inverse but no left inverse.
- You can even have multiple left inverses or multiple right inverses.

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Example

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$ is a right inverse for $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We also have $\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ is a right inverse for $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, since

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

But $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ does not have any left inverses, since for any 3×2 matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$, we have

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 0 \end{bmatrix}$$

Thus, $\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & 0 \end{bmatrix}$ cannot be the identity matrix, since its last column is all zeros.

Note: It turns out that only square matrices can have both a left and a right inverse, and so from this point on we will only concern ourselves with the inverses of square matrices.